

DIFFERENTIATION & FUNCTIONS (Q 6, 7 & 8, PAPER 1)**2010**6 (a) Let $h(x) = x^2 + 1$, where $x \in \mathbf{R}$.Write down a value of x for which $h(x) = 50$.(b) Let $g(x) = \frac{1}{x-2}$, where $x \in \mathbf{R}$ and $x \neq 2$.

(i) Copy and complete the following table:

| | | | | | | | | |
|--------|---|----|-----|------|------|-----|---|---|
| x | 0 | 1 | 1.5 | 1.75 | 2.25 | 2.5 | 3 | 4 |
| $g(x)$ | | -1 | | -4 | | 2 | | |

(ii) Draw the graph of the function g in the domain $0 \leq x \leq 4$.(c) Let $f(x) = x - \frac{5}{x}$, where $x \in \mathbf{R}$ and $x \neq 0$.(i) Find $f'(x)$, the derivative of $f(x)$.

(ii) Find the co-ordinates of the two points at which the tangent to the curve is parallel to the line

SOLUTION

6 (a) $h(x) = x^2 + 1$

$$h(x) = 50 \Rightarrow x^2 + 1 = 50$$

$$x^2 = 49$$

$$x = \pm\sqrt{49} = \pm 7$$

6 (b) (i)

$$x = 0 : g(0) = \frac{1}{(0)-2} = -\frac{1}{2} = -0.5$$

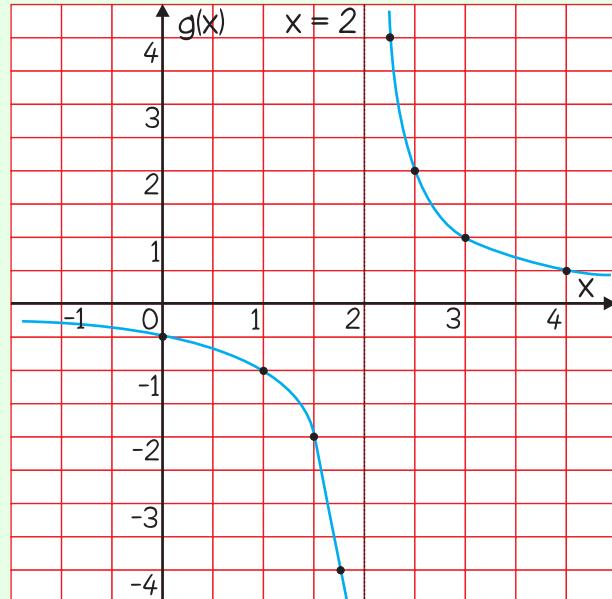
$$x = 1.5 : g(1.5) = \frac{1}{(1.5)-2} = -\frac{1}{0.5} = -2$$

$$x = 2.25 : g(2.25) = \frac{1}{(2.25)-2} = \frac{1}{0.25} = 4$$

$$x = 3 : g(3) = \frac{1}{(3)-2} = \frac{1}{1} = 1$$

$$x = 4 : g(4) = \frac{1}{(4)-2} = \frac{1}{2} = 0.5$$

| | | | | | | | | |
|--------|------|----|-----|------|------|-----|---|-----|
| x | 0 | 1 | 1.5 | 1.75 | 2.25 | 2.5 | 3 | 4 |
| $g(x)$ | -0.5 | -1 | -2 | -4 | 4 | 2 | 1 | 0.5 |

6 (b) (ii)**6 (c) (i)**

$$f(x) = x - \frac{5}{x} = x - 5x^{-1}$$

$$f'(x) = 1 + 5x^{-2} = 1 + \frac{5}{x^2}$$

$$y = x^n \Rightarrow \frac{dy}{dx} = nx^{n-1}$$

$$a^{-n} = \frac{1}{a^n} \text{ ...Power Rule No. 4}$$

6 (c) (ii)

$$y = 6x \Rightarrow \frac{dy}{dx} = 6 \text{ [Differentiate the equation of the line to find its slope.]}$$

$$1 + \frac{5}{x^2} = 6 \text{ [Parallel lines have the same slope. Therefore the slope of the tangents is 6.]}$$

$$\frac{5}{x^2} = 5 \Rightarrow x^2 = 1$$

$$\therefore x = \pm\sqrt{1} = \pm 1$$

$$x = 1: f(1) = (1) - \frac{5}{(1)} = 1 - 5 = -4 \Rightarrow (1, -4) \text{ is a point of contact.}$$

$$x = -1: f(-1) = (-1) - \frac{5}{(-1)} = -1 + 5 = 4 \Rightarrow (-1, 4) \text{ is a point of contact.}$$

7 (a) Differentiate $x^2 - 6x + 1$ with respect to x .

(b) (i) Differentiate $5 - 3x$ with respect to x from first principles.

(ii) Given that $y = (x^2 - 4)(3x - 1)$, find the value of $\frac{dy}{dx}$ when $x = 2$.

(c) The speed, v , of an object at time t is given by

$$v = 96 + 40t - 4t^2$$

where t is in seconds and v is in metres per second.

(i) At what times will the speed of the object be 96 metres per second?

(ii) What will the acceleration of the object be at $t = 2.5$ seconds?

(iii) At what value of t will the acceleration become negative?

SOLUTION

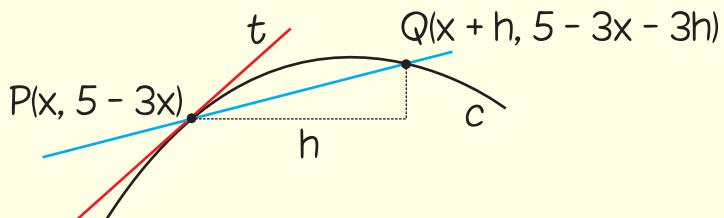
7 (a)

$$y = x^2 - 6x + 1$$

$$\frac{dy}{dx} = 2x - 6$$

$$y = x^n \Rightarrow \frac{dy}{dx} = nx^{n-1}$$

7 (b) (i)



$$y = 5 - 3(x)$$

$$x = x : y = 5 - 3x$$

$$y = 5 - 3(x)$$

$$x = x + h : y = 5 - 3(x + h)$$

$$y = 5 - 3x - 3h$$

$$\begin{array}{c} Q(x + h, 5 - 3x - 3h) \\ \downarrow \\ P(x_1, 5 - 3x_1) \end{array}$$

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$\begin{aligned} \text{Slope of } PQ &= \frac{(5 - 3x - 3h) - (5 - 3x)}{x + h - x} \\ &= \frac{5 - 3x - 3h - 5 + 3x}{h} = \frac{-3h}{h} = -3 \end{aligned}$$

$$\frac{dy}{dx} = \lim_{h \rightarrow 0} (-3) = -3$$

7 (b) (ii)

$$y = (x^2 - 4)(3x - 1)$$

$$\begin{array}{c} \uparrow \\ u \\ \uparrow \\ v \end{array}$$

$$\begin{aligned} u &= x^2 - 4 \Rightarrow \frac{du}{dx} = 2x \\ v &= 3x - 1 \Rightarrow \frac{dv}{dx} = 3 \end{aligned}$$

$$y = uv \Rightarrow \frac{dy}{dx} = u \frac{dv}{dx} + v \frac{du}{dx}$$

$$\begin{aligned} \frac{dy}{dx} &= (x^2 - 4)(3) + (3x - 1)(2x) \\ &= 3x^2 - 12 + 6x^2 - 2x \\ &= 9x^2 - 2x - 12 \end{aligned}$$

$$\left(\frac{dy}{dx} \right)_{x=2} = 9(2)^2 - 2(2) - 12 = 36 - 4 - 12 = 20$$

7 (c) (i)

$$v = 96 \text{ ms}^{-1}$$

$$(v) = 96 + 40t - 4t^2$$

$$96 = 96 + 40t - 4t^2$$

$$0 = 40t - 4t^2$$

$$0 = 10t - t^2 = t(10 - t)$$

$$\therefore t = 0, 10 \text{ s}$$

7 (c) (ii)

$$a = \frac{dv}{dt} = 40 - 8t$$

$$t = 2.5 \text{ s}$$

$$a = 40 - 8(2.5) = 40 - 20 = 20 \text{ ms}^{-2}$$

$$a = \frac{\text{Change in velocity}}{\text{Change in time}} = \frac{dv}{dt}$$

7 (c) (iii)

Solve the inequality $a < 0$.

$$a < 0 \Rightarrow 40 - 8t < 0$$

$$-8t < -40$$

$$t > \frac{40}{8}$$

$$t > 5 \text{ s}$$

8. Let $f(x) = x^3 - 3x + 1$, where $x \in \mathbf{R}$.

- Find $f(-3)$, $f(-2)$, $f(0)$, $f(2)$ and $f(3)$.
- Find $f'(x)$, the derivative of $f(x)$.
- Find the co-ordinates of the local maximum point and of the local minimum point of the curve $y = f(x)$.
- Draw the graph of the function f in the domain $-3 \leq x \leq 3$.
- Find the range of values of k for which the equation

$$x^3 - 3x + 1 = k$$

has three real solutions (roots).

SOLUTION

8 (i)

$$f(x) = (x)^3 - 3(x) + 1$$

$$f(-3) = (-3)^3 - 3(-3) + 1 = -27 + 9 + 1 = -17$$

$$f(-2) = (-2)^3 - 3(-2) + 1 = -8 + 6 + 1 = -1$$

$$f(0) = (0)^3 - 3(0) + 1 = 0 + 0 + 1 = 1$$

$$f(2) = (2)^3 - 3(2) + 1 = 8 - 6 + 1 = 3$$

$$f(3) = (3)^3 - 3(3) + 1 = 27 - 9 + 1 = 19$$

8 (ii)

$$f(x) = x^3 - 3x + 1$$

$$f'(x) = 3x^2 - 3$$

$$y = x^n \Rightarrow \frac{dy}{dx} = nx^{n-1}$$

8 (iii)

$$f'(x) = 0 \Rightarrow 3x^2 - 3 = 0$$

$$x^2 - 1 = 0$$

$$x^2 = 1$$

$$x = \pm\sqrt{1} = \pm 1$$

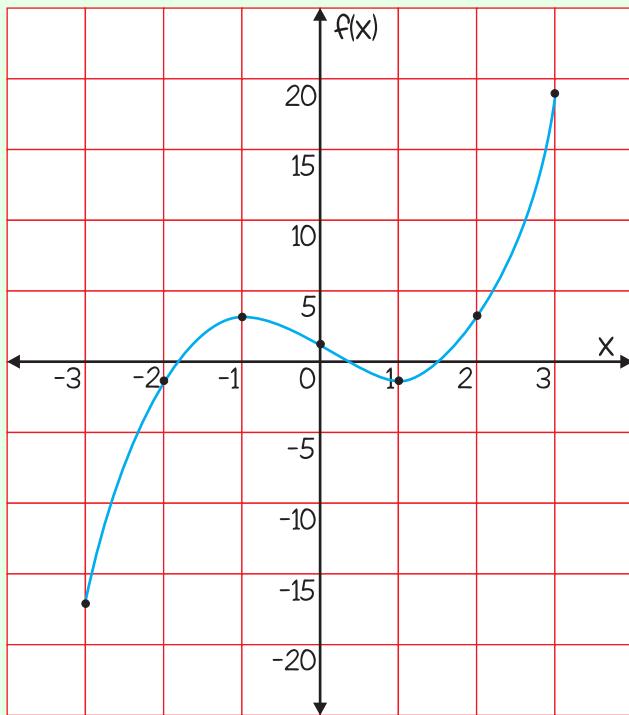
$$\text{Turning Point: } \frac{dy}{dx} = 0$$

$x = 1$: $f(1) = (1)^3 - 3(1) + 1 = 1 - 3 + 1 = -1 \Rightarrow (1, -1)$ is a turning point.

$x = -1$: $f(-1) = (-1)^3 - 3(-1) + 1 = -1 + 3 + 1 = 3 \Rightarrow (-1, 3)$ is a turning point.

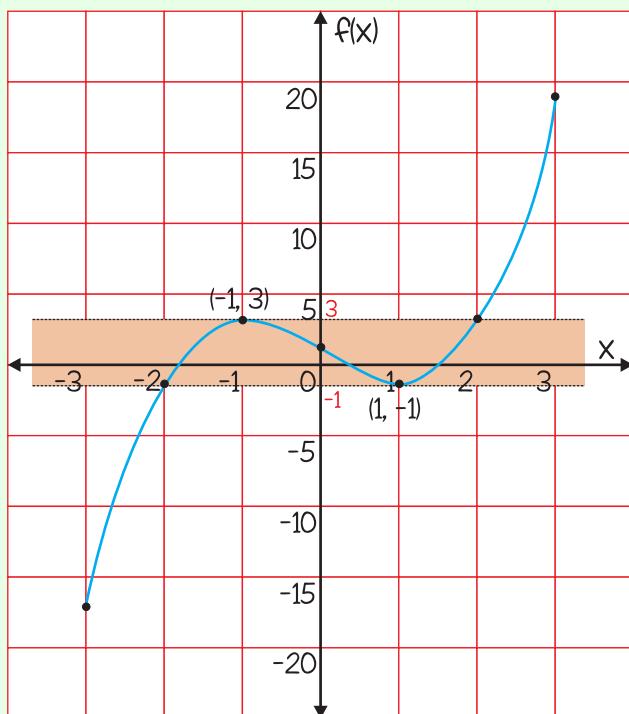
Local maximum $(-1, 3)$ [$(-1, 3)$ is the local maximum as its y coordinate is greater than Local minimum $(1, -1)$ the y coordinate of the other point.]

8 (iv)



| x | $f(x)$ |
|-----|--------|
| -3 | -17 |
| -2 | -1 |
| -1 | 5 |
| 0 | 1 |
| 1 | -1 |
| 2 | 3 |
| 3 | 19 |

8 (v)

3 solutions: $-1 < k < 3$

[Any line drawn through the $f(x)$ axis will cut the graph 3 times anywhere between the maximum and minimum points.]