2009

6 (a) Let $g(x) = 4 - kx$.
Given that $g(-5) = 34$, find the value of k.

(b) Let $h(x) = x(1 - x^2)$, where $x \in \mathbf{R}$.
(i) Verify that $h(3) + h(-3) = 0$.

(ii) Find the values of x for which $h'(x) = -11$, where $h'(x)$ is the derivative of $h(x)$.

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(c) Let $f(x) = x^3 - 6x^2 + 9x - 3$, where $x \in \mathbf{R}$.
(i) Find the co-ordinates of the local maximum point and of the local minimum point of the curve $y = f(x)$.

(ii) Draw the graph of the function f in the domain $0 \le x \le 4$.

(iii) Use your graph to estimate the range of values of x for which $x < 3$ and $f(x) \ge 0$.

SOLUTION

6 (a)

$g(x) = 4 - kx$

$g(-5) = 34 \Rightarrow 4 - k(-5) = 34$

$4 + 5k = 34$

$5k = 30$

$k = 6$

6 (b) (i)

$h(x) = x(1 - x^2)$

$h(3) = (3)(1 - (3)^2) = 3(1 - 9) = 3(-8) = -24$

$h(-3) = (-3)(1 - (-3)^2) = -3(1 - 9) = -3(-8) = 24$

$h(3) + h(-3) = -24 + 24 = 0$

6 (b) (ii)

$h(x) = x(1 - x^2) = x - x^3$

$h'(x) = 1 - 3x^2$

$h'(x) = -11 \Rightarrow 1 - 3x^2 = -11$

$12 = 3x^2$

$4 = x^2$

$x^2 = \pm\sqrt{4} = \pm 2$

$$\boxed{y = x^n \Rightarrow \frac{dy}{dx} = nx^{n-1}}$$

6 (c) (i)

$f(x)=x^3-6x^2+9x-3$
$f'(x)=3x^2-12x+9$

$$y=x^n \Rightarrow \frac{dy}{dx}=nx^{n-1}$$

$f'(x)=0 \Rightarrow 3x^2-12x+9=0$
$x^2-4x+3=0$
$(x-1)(x-3)=0$
$\therefore x=1, 3$

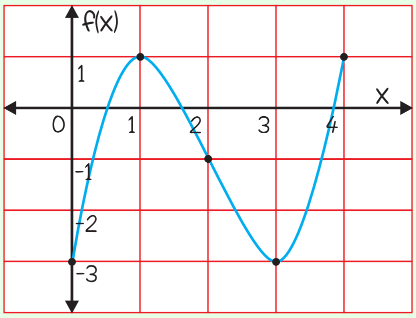
Turning Point: $\frac{dy}{dx}=0$

$x=1: f(1)=(1)^3-6(1)^2+9(1)-3=1-6+9-3=1 \Rightarrow (1, 1)$ is a turning point.
$x=3: f(3)=(3)^3-6(3)^2+9(3)-3=27-54+27-3=-3 \Rightarrow (3, -3)$ is a turning point.

(1, 1) is a local maximum point,
(3, –3) is a local minimum point.

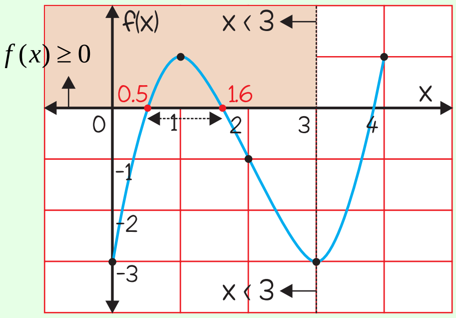
[(1, 1) is the local maximum as its y coordinate is greater than the y coordinate of the other point.]

6 (c) (ii)



x	$f(x)$
0	–3
1	1
2	–1
3	–3
4	1

6 (c) (iii)



x less than 3 is the region left of the line $x = 3$. $f(x)$ greater than or equal to zero is the region above and including the x axis. The shaded area is the area of overlap of these two regions. You can see the part of the graph that is in this shaded region. It exists for values of x between and including 0.5 and 1.6.

Answer: $0.5 \leq x \leq 1.6$

7 (a) Differentiate $3x^5 - 7x^2 + 9x$ with respect to x.

(b) (i) Given that $y = (x^2 - 4x)^5$, find the value of $\frac{dy}{dx}$ when $x = 2$.

(ii) Differentiate $\frac{x^2 - 1}{x^2 + 1}$ with respect to x.

Write your answer in the form $\frac{kx}{(x^2+1)^n}$, where $k, n \in \mathbf{N}$.

(c) A ball is fired straight up in the air.
The height, h metres, of the ball above the ground is given by

$$h = 30t - 5t^2$$

where t is the time in seconds after the ball was fired.

(i) After how many seconds does the ball hit the ground?

(ii) Find the speed of the ball after 2 seconds.

(iii) Find the maximum height reached by the ball.

SOLUTION

7 (a)

$$y = 3x^5 - 7x^2 + 9x$$

$$\frac{dy}{dx} = 15x^4 - 14x + 9$$

$$y = x^n \Rightarrow \frac{dy}{dx} = nx^{n-1}$$

7 (b) (i)

$$y = (x^2 - 4x)^5$$

$$\frac{dy}{dx} = 5(x^2 - 4x)^4(2x - 4)$$

Move the power down in front of the bracket.
Take one away from the power.
Multiply by the differentiation of the inside of the bracket.

$$\left(\frac{dy}{dx}\right)_{x=2} = 5((2)^2 - 4(2))^4(2(2) - 4)$$

$$= 5(4 - 8)^4(4 - 4) = 5(-4)^4(0) = 0$$

7 (b) (ii)

$$y = \frac{x^2 - 1}{x^2 + 1} \quad \begin{matrix} \leftarrow u \\ \leftarrow v \end{matrix}$$

$$y = \frac{u}{v} \Rightarrow \frac{dy}{dx} = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$$

$$u = x^2 - 1 \Rightarrow \frac{du}{dx} = 2x$$
$$v = x^2 + 1 \Rightarrow \frac{dv}{dx} = 2x$$

$$\frac{dy}{dx} = \frac{(x^2+1)(2x) - (x^2-1)(2x)}{(x^2+1)} = \frac{2x^3 + 2x - 2x^3 + 2x}{(x^2+1)}$$
$$= \frac{4x}{(x^2+1)}$$

7 (c) (i)

$h = 0$ [When the ball hits the ground it height h is 0 metres.]

$$h = 30t - 5t^2 \Rightarrow 30t - 5t^2 = 0$$
$$6t - t^2 = 0$$
$$t(6 - t) = 0$$
$$\therefore t = 0,\ 6$$

[Ignore t = 0 s as this is the situation when the ball is fired from the ground.]

Answer: t = 6 s

7 (c) (ii)

$$v = \frac{dh}{dt} = 30 - 10t$$
$$t = 2 : v = 30 - 10(2) = 30 - 20 = 10\,\text{m s}^{-1}$$

$$v = \frac{\text{Change in distance}}{\text{Change in time}} = \frac{ds}{dt}$$

7 (c) (iii)

$$v = 30 - 10t$$
$$v = 0 \Rightarrow 30 - 10t = 0$$
$$30 = 10t$$
$$t = 3\text{ s}$$

[Firstly, you need to find out how long it takes the ball to reach its greatest height. At its greatest height, its velocity v is zero.]

$$h = 30t - 5t^2$$
$$t = 3 : h = 30(3) - 5(3)^2 = 90 - 45 = 45\text{ m}$$

[It takes 3 seconds for the ball to reach its greatest height. Find this height h by replacing t by 3 in the height equation.]

8 (a) Let $g(x) = 2(6 - 3x)$, where $x \in \mathbf{R}$.
Find the value of x for which $g(x) = 0$.

(b) Differentiate $2x^2 - 5x$ with respect to x from first principles.

(c) Let $f(x) = \dfrac{1}{x+1}$, $x \in \mathbf{R}$, $x \neq -1$.

(i) Find $f'(x)$, the derivative of $f(x)$.

(ii) Find the two values of x at which the slope of the tangent to the curve $y = f(x)$ is -1.

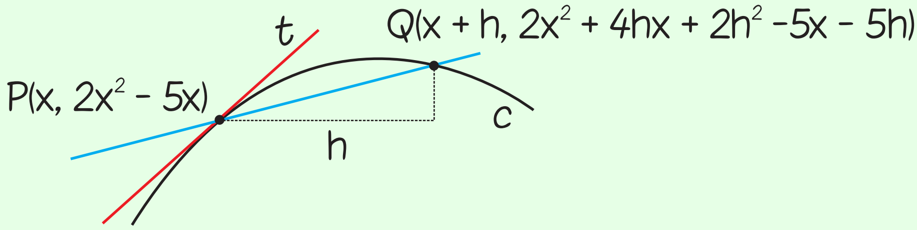
(iii) One of these tangents intersects the positive y-axis.
Find the equation of this tangent.

SOLUTION

8 (a)

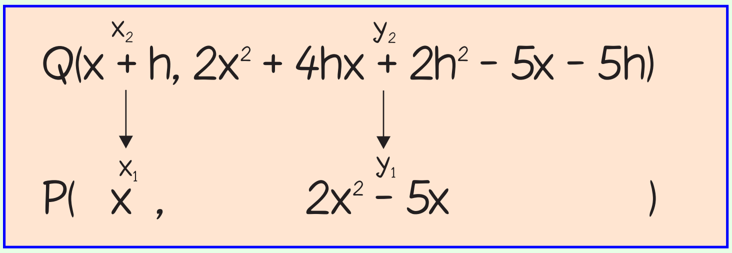
$$g(x) = 2(6-3x)$$
$$g(x) = 0 \Rightarrow 2(6-3x) = 0$$
$$6 - 3x = 0$$
$$6 = 3x$$
$$\therefore x = 2$$

8 (b)



$$y = 5 - 3(x)$$
$$x = x : y = 5 - 3x$$

$$y = 2(x)^2 - 5(x)$$
$$x = x+h : y = 2(x+h)^2 - 5(x+h)$$
$$= 2(x+h)(x+h) - 5(x+h)$$
$$= 2(x^2 + hx + hx + h^2) - 5(x+h)$$
$$= 2(x^2 + 2hx + h^2) - 5(x+h)$$
$$= 2x^2 + 4hx + 2h^2 - 5x - 5h$$



$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Slope of PQ $= \dfrac{(2x^2+4hx+2h^2-5x-5h)-(2x^2-5x)}{x+h-x}$

$= \dfrac{2x^2+4hx+2h^2-5x-5h-2x^2+5x}{x+h-x}$

$= \dfrac{4hx+2h^2-5h}{h} = \dfrac{h(4x+2h-5)}{h}$

$= 4x+2h-5$

$$\frac{dy}{dx} = \lim_{h\to 0}(4x+2h-5) = 4x-5$$

8 (c) (i)

$f(x) = \dfrac{1}{x+1} = (x+1)^{-1}$

$f'(x) = -1(x+1)^{-2}(1) = -\dfrac{1}{(x+1)^2}$

Move the power down in front of the bracket.
Take one away from the power.
Multiply by the differentiation of the inside of the bracket.

$\boxed{a^{-n} = \dfrac{1}{a^n}}$ **...Power Rule No. 4**

8 (c) (ii)

$m = -1 \Rightarrow -\dfrac{1}{(x+1)^2} = -1$

$(x+1)^2 = 1$

$x+1 = \pm\sqrt{1} = \pm 1$

$x+1=1$	$x+1=-1$
$\therefore x=0$	$\therefore x=-2$

Answer: $x = -2, 0$

8 (c) (iii)

You need to find the equation of each tangent.

Equation of t_1:

$x=-2: f(-2) = \dfrac{1}{(-2)+1} = \dfrac{1}{-1} = -1 \Rightarrow (-2, -1)$ is a point of contact.

Point $(x_1, y_1) \to (-2, -1)$, Slope $m = -1$

$\boxed{y - y_1 = m(x - x_1)}$

$(x_1, y_1) \to (-2, -1)$

$y-(-1) = (-1)(x-(-2))$

$y+1 = -1(x+2)$

$y+1 = -x-2$

$x+y+3=0$ [Equation of tangent]

To find the y intercept, put $x = 0$ in the equation of the tangent.

$x=0: (0)+y+3=0 \Rightarrow y=-3$

This tangent cuts the negative y axis.

Equation of t_2:

$$x = 0:\ f(0) = \frac{1}{(0)+1} = \frac{1}{1} = 1 \Rightarrow (0,\ 1) \text{ is a point of contact.}$$

$(x_1,\ y_1) \rightarrow (0,\ 1)$

$y - (1) = (-1)(x - (0))$

$y - 1 = -1(x)$

$y - 1 = -x$

$x + y - 1 = 0$ [Equation of tangent]

To find the y intercept, put $x = 0$ in the equation of the tangent.

$x = 0 : (0) + y - 1 = 0 \Rightarrow y = 1$

This tangent cuts the positive y axis.

Answer: $x + y - 1 = 0$