DIFFERENTIATION & FUNCTIONS (Q 6, 7 & 8, PAPER 1)

2008

6 (a) Let $g(x) = 2x - 5$, where $x \in \mathbf{R}$.
Find the value of x for $g(x) = 19$.

(b) Differentiate $3x^2 + 5$ with respect to x from first principles.

(c) Let $f(x) = \dfrac{x^2 - x}{1 - x^3}$, $x \in \mathbf{R}$, $x \neq 1$.

(i) Find $f'(x)$, the derivative of $f(x)$.

(ii) Show that the tangent to the curve $y = f(x)$ at the point (0, 0) makes an angle of 135° with the positive sense of the x-axis.

SOLUTION

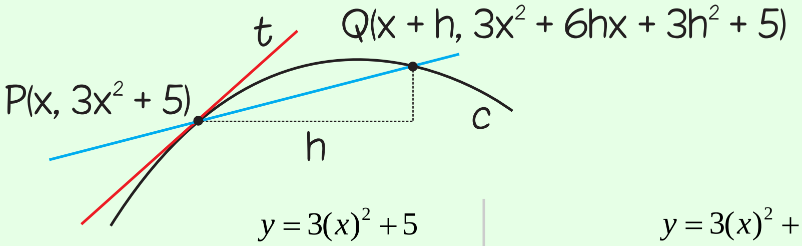
6 (a)

$g(x) = 19 \Rightarrow 2x - 5 = 19$

$\Rightarrow 2x = 24$

$\therefore x = 12$

6 (b)



$$y = 3(x)^2 + 5$$
$$x = x : y = 3x^2 + 5$$

$$y = 3(x)^2 + 5$$
$$\begin{aligned} x = x + h : y &= 3(x+h)^2 + 5 \\ &= 3(x+h)(x+h) + 5 \\ &= 3(x^2 + hx + hx + h^2) + 5 \\ &= 3(x^2 + 2hx + h^2) + 5 \\ &= 3x^2 + 6hx + 3h^2 + 5 \end{aligned}$$

$$Q(\overset{x_2}{x + h},\ \overset{y_2}{3x^2 + 6hx + 3h^2 + 5})$$
$$\downarrow \qquad\qquad \downarrow$$
$$P(\overset{x_1}{x},\ \overset{y_1}{3x^2 + 5})$$

$$\boxed{m = \frac{y_2 - y_1}{x_2 - x_1}}$$

$$\begin{aligned} \text{Slope of } PQ &= \frac{(3x^2 + 6hx + 3h^2 + 5) - (3x^2 + 5)}{x + h - x} \\ &= \frac{3x^2 + 6hx + 3h^2 + 5 - 3x^2 - 5}{x + h - x} \\ &= \frac{6hx + 3h^2}{h} \\ &= \frac{h(6x + 3h)}{h} = 6x + 3h \end{aligned}$$

$$\frac{dy}{dx} = \lim_{h \to 0}(6x + 3h) = 6x$$

6 (c) (i)

$$u = x^2 - x \Rightarrow \frac{du}{dx} = 2x - 1$$
$$v = 1 - x^3 \Rightarrow \frac{dv}{dx} = -3x^2$$

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$\frac{dy}{dx} = f'(x) = \frac{(1-x^3)(2x-1)-(x^2-x)(-3x^2)}{(1-x^3)^2}$$
$$\Rightarrow f'(x) = \frac{2x-1-2x^4+x^3+3x^4-3x^3}{(1-x^3)^2}$$
$$\Rightarrow f'(x) = \frac{x^4-2x^3+2x-1}{(1-x^3)^2}$$

6 (c) (ii)

The slope of the tangent at (0, 0) can be found by calculating $f'(0)$.

$$\left(\frac{dy}{dx}\right)_{x=0} = f'(0) = \frac{(0)^4 - 2(0)^3 + 2(0) - 1}{(1-(0)^3)^2} = \frac{-1}{1} = -1$$

The slope is also the tan of the angle with the positive sense of the *x*-axis.

$\therefore \tan\theta = -1 \Rightarrow \theta = 135^\circ$

7 (a) Differentiate with respect to x

(i) x^7

(ii) $5x - 3x^4$.

(b) (i) Differentiate $(1+3x)(4-x^2)$ with repsect to x.

(ii) Given that $y = (3x^2 - 4x)^8$, find $\frac{dy}{dx}$ when $x = 1$.

(c) A distress flare is tested by firing it vertically upwards from the top of a tower.
The height, h metres, of the flare above the ground is given by

$$h = 20 + 90t - 5t^2$$

where t is the time in seconds from the instant the flare is fired.
The flare is designed to explode 7 seconds after firing.

(i) Find the height above the ground at which the flare explodes.

(ii) Find the speed of the flare at the instant it explodes.

(iii) If the flare failed to explode, find the greatest height above the ground it would reach before falling back down.

SOLUTION

7 (a) (i)

$$y = x^7 \Rightarrow \frac{dy}{dx} = 7x^6$$

$$\boxed{y = x^n \Rightarrow \frac{dy}{dx} = nx^{n-1}}$$

7 (a) (ii)

$$y = 5x - 3x^4 \Rightarrow \frac{dy}{dx} = 5 - 12x^3$$

7 (b) (i)

$$u = (1+3x) \Rightarrow \frac{du}{dx} = 3$$
$$v = (4-x^2) \Rightarrow \frac{dv}{dx} = -2x$$

$$\boxed{\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}}$$

$$\frac{dy}{dx} = (1+3x)(-2x) + (4-x^2)(3)$$

$$\Rightarrow \frac{dy}{dx} = -2x - 6x^2 + 12 - 3x^2$$

$$\therefore \frac{dy}{dx} = 12 - 2x - 9x^2$$

7 (b) (ii)

Move the power down in front of the bracket.
Take one away from the power.
Multiply by the differentiation of the inside of the bracket.

$$y = (3x^2 - 4x)^8$$

$$\Rightarrow \frac{dy}{dx} = 8(3x^2 - 4x)^7(6x - 4)$$

$$\therefore \left(\frac{dy}{dx}\right)_{x=1} = 8(3(1)^2 - 4(1))^7(6(1) - 4) = 8(3-4)^7(6-4)$$

$$\Rightarrow \left(\frac{dy}{dx}\right)_{x=1} = 8(-1)^7(2) = 8(-1)(2)$$

$$\therefore \left(\frac{dy}{dx}\right)_{x=1} = -16$$

7 (c) (i)

The flare explodes after 7 seconds. Let t = 7 s to find the height h.

$$h = 20 + 90t - 5t^2$$

$$\Rightarrow h = 20 + 90(7) - 5(7)^2$$

$$\Rightarrow h = 20 + 630 - 5\times 49$$

$$\Rightarrow h = 20 + 630 - 245$$

$$\therefore h = 405 \text{ m}$$

7 (c) (ii)

Differentiate the height h with respect to time t to find the speed v after 7 seconds.

$$v = \frac{dh}{dt} = 90 - 10t \qquad \boxed{v = \frac{ds}{dt}}$$

$$\Rightarrow \left(\frac{dh}{dt}\right)_{t=7} = 90 - 10(7)$$

$$\therefore v = 90 - 70 = 20 \text{ m/s}$$

7 (c) (iii)

At the greatest height h the velocity of the flare $v = 0$. Put the velocity equal to zero and solve for t.

$$v = 0 \Rightarrow 90 - 10t = 0$$

$$\Rightarrow 90 = 10t$$

$$\therefore t = 9 \text{ s}$$

It takes 9 seconds to reach the greatest height. Let $t = 9$ s in the formula for the height h.

$$h = 20 + 90(9) - 5(9)^2$$

$$\Rightarrow h = 20 + 810 - 5 \times 81$$

$$\Rightarrow h = 20 + 810 - 405$$

$$\therefore h = 425 \text{ m}$$

8 Let $f(x) = x^3 - 9x^2 + 24x - 18$, where $x \in \mathbf{R}$.

(i) Find $f(1)$ and $f(5)$.

(ii) Find $f'(x)$, the derivative of $f(x)$.

(iii) Find the co-ordinates of the local maximum point and of the local minimum point of the curve $y = f(x)$.

(iv) Draw the graph of the function f in the domain $1 \le x \le 5$.

(v) Use your graph to write down the range of values of x for which $f'(x) < 0$.

(vi) The line $y = -3x + c$ is a tangent to the curve $y = f(x)$. Find the value of c.

Solution

8 (i)

$$f(x) = x^3 - 9x^2 + 24x - 18$$

$$\Rightarrow f(1) = (1)^3 - 9(1)^2 + 24(1) - 18 = 1 - 9 + 24 - 18 = -2$$

$$\Rightarrow f(5) = (5)^3 - 9(5)^2 + 24(5) - 18 = 125 - 225 + 120 - 18 = 2$$

8 (ii)

$$f(x) = x^3 - 9x^2 + 24x - 18$$
$$\Rightarrow f'(x) = 3x^2 - 18x + 24$$

$$y = x^n \Rightarrow \frac{dy}{dx} = nx^{n-1}$$

8 (iii)

$$f'(x) = 3x^2 - 18x + 24$$

$$f'(x) = 0 \Rightarrow 3x^2 - 18x + 24 = 0$$
$$\Rightarrow x^2 - 6x + 8 = 0$$
$$\Rightarrow (x-2)(x-4) = 0$$
$$\therefore x = 2,\ 4$$

$$\text{Turning Point} \Rightarrow \frac{dy}{dx} = 0$$

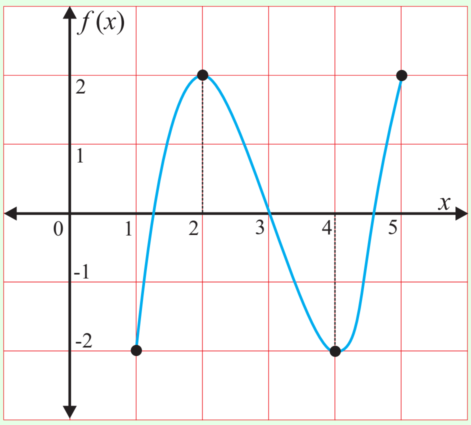
$f(2) = (2)^3 - 9(2)^2 + 24(2) - 18 = 8 - 36 + 48 - 18 = 2 \Rightarrow (2,\ 2)$ is a local maximum.

$f(4) = (4)^3 - 9(4)^2 + 24(4) - 18 = 64 - 144 + 96 - 18 = -2 \Rightarrow (4,\ -2)$ is a local minimum.

[(2, 2) is the local maximum as its y coordinate is greater than the y coordinate of the other point.]

8 (iv)

From part **(i)**, you worked out 2 points: (1, –2) and (5, 2).
From part **(iii)**, you worked out 2 more points:
Local maximum (2, 2) and local minimum (4, –2).



8 (v)

Positive Slope: $\frac{dy}{dx} > 0$

Negative Slope: $\frac{dy}{dx} < 0$

You can see from the graph that it is decreasing (going downhill as you walk from left to right) for values of *x* from 2 to 4.

$f'(x) < 0 : 2 < x < 4$

8 (vi)

Find the slope of the tangent by differentiating its equation.

$y = -3x + c \Rightarrow \frac{dy}{dx} = -3$

Find the point on the curve which has a slope of –3 by putting $f'(x) = -3$ and solving for *x*.

$f'(x) = -3 \Rightarrow 3x^2 - 18x + 24 = -3$

$\Rightarrow 3x^2 - 18x + 27 = 0$

$\Rightarrow x^2 - 6x + 9 = 0$

$\Rightarrow (x-3)(x-3) = 0$

$\therefore x = 3$

Find the co-ordinates of the point of contact with the tangent by finding $f(3)$.

$f(x) = x^3 - 9x^2 + 24x - 18$

$\Rightarrow f(3) = (3)^3 - 9(3)^2 + 24(3) - 18$

$\Rightarrow f(3) = 27 - 81 + 72 - 18 = 0$

$\Rightarrow (3,\ 0)$ is the point of contact between the curve and tangent.

$(3,\ 0) \in y = -3x + c$

$\Rightarrow 0 = -3(3) + c$

$\Rightarrow 0 = -9 + c$

$\therefore c = 9$