DIFFERENTIATION & FUNCTIONS (Q 6, 7 & 8, PAPER 1)

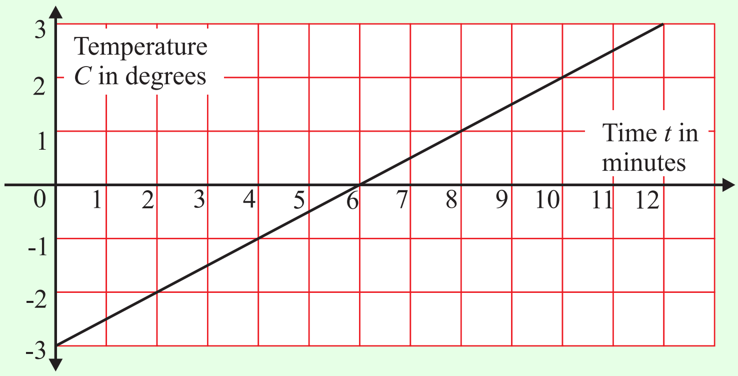
2007

6 (a) Let $g(x) = x^2 - 6x,\ x \in \mathbf{R}$.

(i) Write down $g'(x)$, the derivative of $g(x)$.

(ii) For what value of x is $g'(x) = 0$?

(b) A cold object is placed in a warm room.
Its temperature C degrees after time t minutes is shown in the following graph.



(i) After what time interval is the temperature of the object 0 degrees?

(ii) What is the rise in temperature of the object in the first 10 minutes?

(iii) The relationship between the temperature C and the time t is given by

$$C = \tfrac{1}{2}(t + k).$$

Find the value of k.

(c) Let $f(x) = (5x - 2)^4$ for $x \in \mathbf{R}$.

(i) Find $f'(x)$, the derivative of $f(x)$.

(ii) Find the co-ordinates of the point on the curve $y = f(x)$ at which the slope of the tangent is 20.

SOLUTION

6 (a) (i)

$$\boxed{y = x^n \Rightarrow \frac{dy}{dx} = nx^{n-1}}$$

MULTIPLY BY A CONSTANT RULE: If $y = cu$, where c is a constant and u is a function of x, $\dfrac{dy}{dx} = c \times \dfrac{du}{dx}$.

$g(x) = x^2 - 6x \Rightarrow g'(x) = 2x - 6$

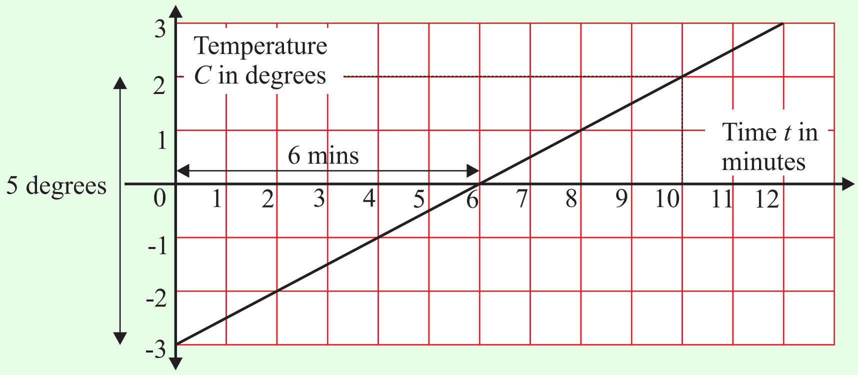
6 (a) (ii)

$g'(x) = 0 \Rightarrow 2x - 6 = 0$

$\Rightarrow 2x = 6 \Rightarrow x = 3$

6 (b) (i)

Look at the graph. After 0 minutes the temperature is -3 degrees.
After 6 minutes the temperature reaches 0 degrees.
Ans: 6 minutes



6 (b) (ii)

Look at the graph. After 0 minutes the temperature is -3 degrees.
After 10 minutes the temperature is 2 degrees.
Therefore, the change in temperature in the first 10 minutes is $2-(-3)=5$ degrees.

6 (b) (iii)

From the graph you can see that $C = 2$ degrees when $t = 10$ minutes.

$$C=\tfrac{1}{2}(t+k)$$
$$\Rightarrow 2=\tfrac{1}{2}(10+k)$$
$$\Rightarrow 4=10+k$$
$$\Rightarrow k=-6$$

6 (c) (i)

$$f(x)=(5x-2)^4$$
$$\Rightarrow f'(x)=4(5x-2)^3(5)$$
$$\Rightarrow f'(x)=20(5x-2)^3$$

Move the power down in front of the bracket.
Take one away from the power.
Multiply by the differentiation of the inside of the bracket.

6 (c) (ii)

Steps: Finding points of contact given the slope, m, of the tangent

1. Find $\dfrac{dy}{dx}$.

2. Put $\dfrac{dy}{dx}=m$ and solve for x.

3. Find the corresponding y values.

1. $f'(x) = 20(5x-2)^3$

2. $f'(x) = 20 \Rightarrow 20(5x-2)^3 = 20$

$\Rightarrow (5x-2)^3 = 1$

$\Rightarrow 5x - 2 = 1$

$\Rightarrow 5x = 3$

$\Rightarrow x = \frac{3}{5}$

3. $f(x) = (5x-2)^4$

$\Rightarrow f(\frac{3}{5}) = (5(\frac{3}{5}) - 2)^4$

$\Rightarrow f(\frac{3}{5}) = (3-2)^4$

$\Rightarrow f(\frac{3}{5}) = (1)^4 = 1$

Point: $(\frac{3}{5}, 1)$

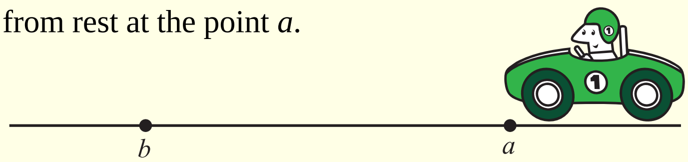
7 (a) Differentiate $6x^4 - 3x^2 + 7x$ with respect to x.

(b) (i) Differentiate $(x^2+9)(4x^3+5)$ with respect to x.

(ii) Given that $y = \dfrac{3x}{2x+3}$, find $\dfrac{dy}{dx}$.

Write your answer in the form $\dfrac{k}{(2x+3)^n}$, where $k, n \in \mathbf{N}$.

(c) A car starts from rest at the point a.



The distance of the car from a, after t seconds, is given by

$$s = 2t^2 + 2t$$

where s is in metres.

(i) Find the speed of the car after 2 seconds.

(ii) Find the acceleration of the car.

(iii) The distance from a to the point b is 24 metres. After how many seconds does the car reach the point b?

Solution

7 (a)

$$\boxed{y = x^n \Rightarrow \frac{dy}{dx} = nx^{n-1}}$$

Multiply by a constant rule: If $y = cu$, where c is a constant and u is a function of x, $\dfrac{dy}{dx} = c \times \dfrac{du}{dx}$.

$$y = 6x^4 - 3x^2 + 7x \Rightarrow \frac{dy}{dx} = 24x^3 - 6x + 7$$

7 (b) (i)

$$\boxed{\frac{dy}{dx}=u\frac{dv}{dx}+v\frac{du}{dx}}$$

$$\frac{dy}{dx}=u\frac{dv}{dx}+v\frac{du}{dx}$$
$$\Rightarrow \frac{dy}{dx}=(x^2+9)(12x^2)+(4x^3+5)(2x)$$
$$\Rightarrow \frac{dy}{dx}=12x^4+108x^2+8x^4+10x$$
$$\Rightarrow \frac{dy}{dx}=20x^4+108x^2+10x$$

$$u=x^2+9\Rightarrow \frac{du}{dx}=2x+0=2x$$
$$v=4x^3+5\Rightarrow \frac{dv}{dx}=4\times 3x^2+0=12x^2$$

7 (b) (ii)

$$\boxed{\frac{dy}{dx}=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}}$$

$$\frac{dy}{dx}=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$$
$$\Rightarrow \frac{dy}{dx}=\frac{(2x+3)3-(3x)2}{(2x+3)^2}$$
$$\Rightarrow \frac{dy}{dx}=\frac{6x+9-6x}{(2x+3)^2}$$
$$\Rightarrow \frac{dy}{dx}=\frac{9}{(2x+3)^2}$$

$$u=3x\Rightarrow \frac{du}{dx}=3$$
$$v=2x+3\Rightarrow \frac{dv}{dx}=2+0=2$$

7 (c) (i)

Draw up an *s*, *v*, *a* table as shown on the right.

Put t = 2 s into the velocity equation.

$$\boxed{v=\frac{ds}{dt}}$$

$$v=4t+2$$
$$\Rightarrow v=4(2)+2=8+2=10 \text{ m/s}$$

$$\boxed{a=\frac{dv}{dt}}$$

$$s=2t^2+2t$$
$$v=\frac{ds}{dt}=2\times 2t+2=4t+2$$
$$a=\frac{dv}{dt}=4$$

7 (c) (ii)

a = 4 m/s^2

7 (c) (iii)

Put the distance equation equal to 24 m and solve for t.

$$s=24\Rightarrow 2t^2+2t=24$$
$$\Rightarrow 2t^2+2t-24=0$$
$$\Rightarrow t^2+t-12=0$$
$$\Rightarrow (t+4)(t-3)=0$$
$$\Rightarrow t=-4,\ 3$$

Ignore the negative solution. Therefore t = 3 s.

8 (a) Let $f(x)=\frac{1}{4}(6-2x)$ for $x\in\mathbf{R}$. Evaluate $f(5)$.

(b) Differentiate x^2-3x with respect to x from first principles.

(c) Let $f(x)=\dfrac{1}{x+7}$, $x\in\mathbf{R}$, $x\neq -7$.

(i) Given that $f(k)=1$, find k.

(ii) Find $f'(x)$, the derivative of $f(x)$.

(iii) Show that the curve $y=f(x)$ has no turning points.

SOLUTION

8 (a)

$f(x)=\frac{1}{4}(6-2x)$

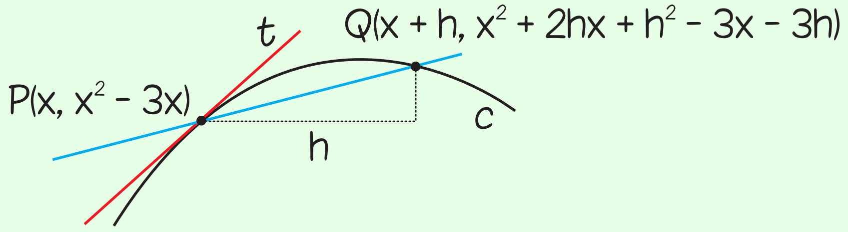
$\Rightarrow f(5)=\frac{1}{4}(6-2(5))$

$\Rightarrow f(5)=\frac{1}{4}(6-10)$

$\Rightarrow f(5)=\frac{1}{4}(-4)$

$\Rightarrow f(5)=-1$

8 (b)



$$y=(x)^2-3(x)$$
$$x=x: y=x^2-3x$$

$$y=(x)^2-3(x)$$
$$\begin{aligned} x=x+h: y&=(x+h)^2-3(x+h)\\ &=(x+h)(x+h)-3(x+h)\\ &=(x^2+hx+hx+h^2)-3(x+h)\\ &=(x^2+2hx+h^2)-3(x+h)\\ &=x^2+2hx+h^2-3x-3h \end{aligned}$$

$$\begin{array}{lll} Q(& \overset{x_2}{x+h}, & \overset{y_2}{x^2+2hx+h^2-3x-3h}) \\ & \downarrow & \downarrow \\ P(& \underset{}{\overset{x_1}{x}}, & \overset{y_1}{x^2-3x}) \end{array}$$

$$m=\frac{y_2-y_1}{x_2-x_1}$$

$$\text{Slope of } PQ = \frac{(x^2+2hx+h^2-3x-3h)-(x^2-3x)}{x+h-x}$$
$$= \frac{x^2+2hx+h^2-3x-3h-x^2+3x}{x+h-x}$$
$$= \frac{2hx+h^2-3h}{h}$$
$$= \frac{h(2x+h-3)}{h} = 2x+h-3$$

$$\frac{dy}{dx} = \lim_{h\to 0}(2x+h-3) = 2x-3$$

8 (c) (i)

$$f(x) = \frac{1}{x+7}$$

$f(k)=1 \Rightarrow \dfrac{1}{(k+7)} = 1$ [Multiply across by (k + 7).]

$\Rightarrow k+7=1$

$\Rightarrow k=-6$

8 (c) (ii)

$$f(x) = \frac{1}{(x+7)} = (x+7)^{-1}$$
$$\Rightarrow f'(x) = -1(x+7)^{-2}(1)$$
$$\Rightarrow f'(x) = -\frac{1}{(x+7)^2}$$

8 (c) (iii)

$f'(x)=0$

$\Rightarrow -\dfrac{1}{(x+7)^2} = 0$ [Multiply across by $(x+7)^2$.]

$\boxed{\text{Turning Point} \Rightarrow \dfrac{dy}{dx} = 0}$

$\Rightarrow -1 = 0$ [This equation is nonsense and has no solutions.]

As there are no solutions for $f'(x)=0,$ the curve $y = f(x)$ has no turning points.