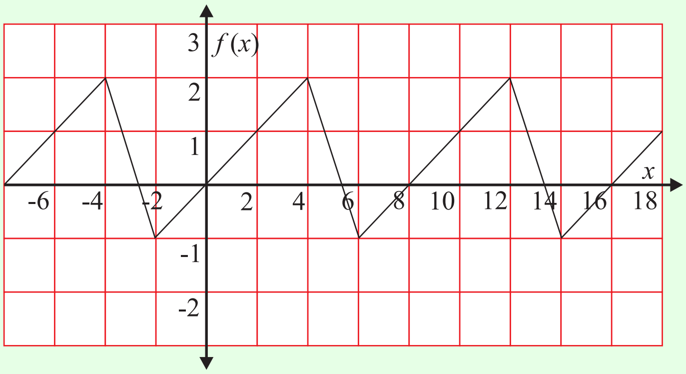
DIFFERENTIATION & FUNCTIONS (Q 6, 7 & 8, PAPER 1)

2006

6 (a) $f : x \to f(x)$ is a periodic function defined for $x \in \mathbf{R}$.
The period is as indicated in the diagram.



(i) Write down the period and the range of the function.

(ii) Find $f(44)$.

(b) The temperature, C, in degrees Celsius, of a liquid in an insulated container is related to time t, in hours, by

$$C = 86 - 6t.$$

(i) Draw the straight line graph of this relation, putting t on the horizontal axis, for $0 \le t \le 8$.

(ii) Use your graph to estimate the temperature when $t = 5.5$ hours.

(iii) Use your graph to estimate the time it takes for the temperature to fall from 80 degrees to 60 degrees.

(c) Let $f(x) = 3 + 8x - 2x^2$, $x \in \mathbf{R}$.

(i) Find the co-ordinates of the point at which the curve $y = f(x)$ cuts the y-axis.

(ii) Find the value of x for which $f(x)$ is a maximum.

(iii) For what range of values of x is $f'(x) > 4$?

SOLUTION

6 (a) (i)

Period = 8

Range = $[-1, 2]$

PERIOD: Length of the wave along the x-axis before it repeats itself.

RANGE: The interval between the lowest y value and the highest y value.

6 (a) (ii)

$f(44) = f(4) = 2$

The value of the function at any value of x can be worked out from the first wave by dividing the value of x by the period and finding the remainder.

$$f(x) = f(\text{Remainder})$$

6 (b) (i)

$C = 86 - 6t$

As these graphs are straight lines, you only need to plot the first and last points in the domain.

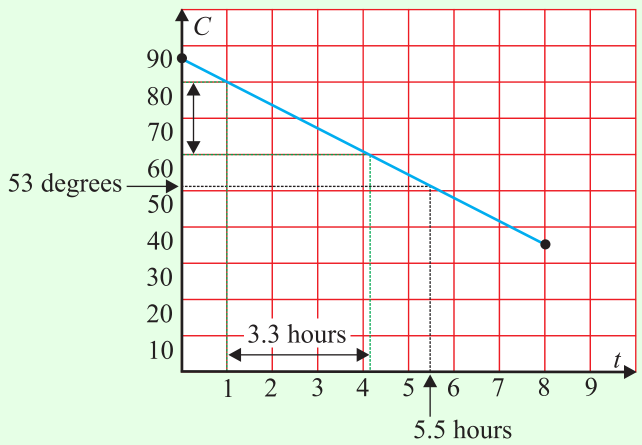
t	0	8
86	86	86
$-6t$	0	-48
C	50	10

OR

$t = 0: C = 86 - 6(0) = 86 - 0 = 86$

$t = 8: C = 86 - 6(8) = 86 - 48 = 34$

The two end points of your straight line are (0, 86) and (8, 34).



6 (b) (ii)

Go to 5.5 hours on the horizontal axis. Go straight up until you meet the straight line graph and then go out to the vertical axis. Read off the temperature.

Ans: C = 53 degrees

6 (b) (iii)

Draw lines from 80 degrees and 60 degrees on the vertical axis to the graph of the straight line and go straight down to the horizontal axis. Measure the time between both points.

Ans: $t = 4.3 \text{ hours} - 1 \text{ hour} = 3.3 \text{ hours}$

6 (c) (i)

$f(x) = 3 + 8x - 2x^2$

Graph cuts y axis: Put $x = 0$.

$\Rightarrow f(x) = 3 + 8(0) - 2(0)^2 = 3$

The graph cuts the y-axis at (0, 3).

6 (c) (ii)

$f(x) = 3 + 8x - 2x^2$

$\Rightarrow f'(x) = 0 + 8 - 2 \times 2x$

$\Rightarrow f'(x) = 8 - 4x$

To find the value of x of the turning point which you are told is a maximum, put $f'(x) = 0$.

Turning Point: $\dfrac{dy}{dx} = 0$

$f'(x) = 0 \Rightarrow 8 - 4x = 0$

$\Rightarrow 2 - x = 0$

$\Rightarrow x = 2$

6 (c) (iii)

$f'(x) > 0 \Rightarrow 8 - 4x > 4$

$\Rightarrow 2 - x > 1$

$\Rightarrow -x > -1$ [Multiply across by a negative number. Remember to reverse the inequality.]

$\Rightarrow x < 1$

7 (a) Differentiate $5x^3 - 4x + 7$ with respect to x.

(b) (i) Differentiate $\dfrac{x^2 - 1}{x + 1}$ with respect to x and write your answer in its simplest form.

(ii) Given that $y = (5 - x^2)^3$, find $\dfrac{dy}{dx}$ when $x = 2$.

(c) A missile is fired straight up in the air. The height, h metres, of the missile above the firing position is given by

$$h = t(200 - 5t)$$

where t is the time in seconds from the instant the missile was fired.

(i) Find the speed of the missile after 10 seconds.

(ii) Find the acceleration of the missile.

(iii) One second before reaching its greatest possible height, the missile strikes a target. Find the height of the target.

Solution

7 (a)

$$\boxed{y = x^n \Rightarrow \frac{dy}{dx} = nx^{n-1}}$$

Constant Rule: If y = Constant $\Rightarrow \dfrac{dy}{dx} = 0$

Multiply by a constant rule: If $y = cu$, where c is a constant and u is a function of x, $\dfrac{dy}{dx} = c \times \dfrac{du}{dx}$.

$y = 5x^3 - 4x + 7$

$\Rightarrow \dfrac{dy}{dx} = 5 \times 3x^2 - 4 \times 1 + 0$

$\Rightarrow \dfrac{dy}{dx} = 15x^2 - 4$

7 (b) (i)

$$u = x^2 - 1 \Rightarrow \frac{du}{dx} = 2x - 0 = 2x$$
$$v = x + 1 \Rightarrow \frac{dv}{dx} = 1 + 0 = 1$$

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$y = \frac{x^2-1}{x+1} \Rightarrow \frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2} = \frac{(x+1)(2x)-(x^2-1)(1)}{(x+1)^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{2x^2+2x-x^2+1}{(x+1)^2}$$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{x^2+2x+1}{(x+1)^2}$ [You can factorise the top: $x^2+2x+1=(x+1)(x+1)=(x+1)^2$.]

$$\Rightarrow \frac{dy}{dx} = \frac{(x+1)^2}{(x+1)^2} = 1$$

7 (b) (ii)

Move the power down in front of the bracket.
Take one away from the power.
Multiply by the differentiation of the inside of the bracket.

$$u = (5-x^2) \Rightarrow \frac{du}{dx} = 0 - 2x = -2x$$

$$y = (5-x^2)^3 \Rightarrow \frac{dy}{dx} = 3(5-x^2)^2 \times (-2x)$$

$\Rightarrow \dfrac{dy}{dx} = -6x(5-x^2)^2$ [Replace x by 2.]

$$\Rightarrow \left(\frac{dy}{dx}\right)_{x=2} = -6(2)(5-(2)^2)^2$$

$$= -12(5-4)^2 = -12(1)^2 = -12(1) = -12$$

7 (c) (i)

Draw up a s, v, a table as shown on the right.

$$v = \frac{ds}{dt}$$
$$a = \frac{dv}{dt}$$

$$v = 200 - 10t = 200 - 10(10)$$
$$= 200 - 100 = 100 \text{ m/s}$$

7 (c) (ii)

$$a = -10 \text{ m/s}^2$$

$$h = t(200-5t) = 200t - 5t^2$$
$$v = \frac{dh}{dt} = 200 - 25t$$
$$a = \frac{dv}{dt} = -25$$

7 (c) (iii)

At its greatest possible height, its velocity is zero.
Put v = 0 m/s and solve for t.

$$v = 200 - 10t \Rightarrow 0 = 200 - 10t$$
$$\Rightarrow 0 = 20 - t \Rightarrow t = 20 \text{ s}$$

One second before reaching its greatest height (t = 19 s), it hits a target.

CONT...

Find the height after 19 s.

$\therefore h = t(200-5t) = (19)(200-5(19))$

$\Rightarrow h = (19)(200-95)$

$\Rightarrow h = (19)(105) = 1995$ m

8 (a) Let $g(x) = \dfrac{3}{x+1}$, $x \in \mathbf{R}$, $x \neq -1$.

Evaluate $g(0.5) - g(-0.5)$.

(b) Let $h(x) = x^2 + 2x - 1$, $x \in \mathbf{R}$.

(i) Simplify $h(x-5)$.

(ii) Find the value of x for which $h(x-5) = h(x) - 5$.

(c) Let $f(x) = \dfrac{1}{x-2}$, $x \in \mathbf{R}$, $x \neq 2$.

(i) Find $f'(x)$, the derivative of $f(x)$.

(ii) Find the values of x for which $f'(x) = -1$.

(iii) Find the co-ordinates of the two points on the curve $y = f(x)$ at which the slope of the tangent is –1.

SOLUTION

8 (a)

$g(x) = \dfrac{3}{x+1}$

$\therefore g(0.5) = \dfrac{3}{0.5+1} = \dfrac{3}{1.5} = 2$

$\therefore g(-0.5) = \dfrac{3}{-0.5+1} = \dfrac{3}{0.5} = 6$

$\Rightarrow g(0.5) - g(-0.5) = 2 - 6 = -4$

8 (b) (i)

$h(x) = x^2 + 2x - 1$ [Replace x by $(x-5)$.]

$\Rightarrow h(x-5) = (x-5)^2 + 2(x-5) - 1$

$= x^2 - 10x + 25 + 2x - 10 - 1$

$= x^2 - 8x + 14$

8 (b) (ii)

$h(x-5) = h(x) - 5$

$\Rightarrow x^2 - 8x + 14 = x^2 + 2x - 1 - 5$

$\Rightarrow -8x + 14 = 2x - 6$

$\Rightarrow 14 + 6 = 2x + 8x$

$\Rightarrow 20 = 10x$

$\therefore x = 2$

8 (c) (i)

$$f(x)=\frac{1}{x-2}=(x-2)^{-1}$$

$$\Rightarrow f'(x)=-1(x-2)^{-2}(1)=-\frac{1}{(x-2)^2}$$

8 (c) (ii)

Put $f'(x)=-1$ and solve for *x*.

$$f'(x)=-1\Rightarrow -\frac{1}{(x-2)^2}=-1$$

$$\Rightarrow 1=1(x-2)^2$$

$$\Rightarrow \pm 1=(x-2)$$

$$\therefore 1=x-2\Rightarrow x=3$$

$$\therefore -1=x-2\Rightarrow x=1$$

8 (c) (iii)

You have already found the values of *x* for which the slope of the tangents are -1. To find their corresponding *y* values (or $f(x)$ values) substitute these values of *x* into the original equation.

$$x=1:\ f(x)=\frac{1}{x-2}\Rightarrow f(1)=\frac{1}{(1)-2}=\frac{1}{-1}=-1$$

$$x=3:\ f(x)=\frac{1}{x-2}\Rightarrow f(3)=\frac{1}{(3)-2}=\frac{1}{1}=1$$

The co-ordinates of the points are: $(1,\ -1),\ (3,\ 1)$.