2005

6 (a) Let $g(x) = \dfrac{x+5}{2}$, $x \in \mathbf{R}$.

Find $g(0) + g(2)$.

(b) Differentiate $3x - x^2$ with respect to x from first principles.

(c) Let $f(x) = x^2 + px + 10$, $x \in \mathbf{R}$, where $p \in \mathbf{Z}$.

(i) Find $f'(x)$, the derivative of $f(x)$.

(ii) The minimum value of $f(x)$ is at $x = 3$. Find the value of p.

(iii) Find the equation of the tangent to $f(x)$ at the point (0, 10).

SOLUTION

6 (a)

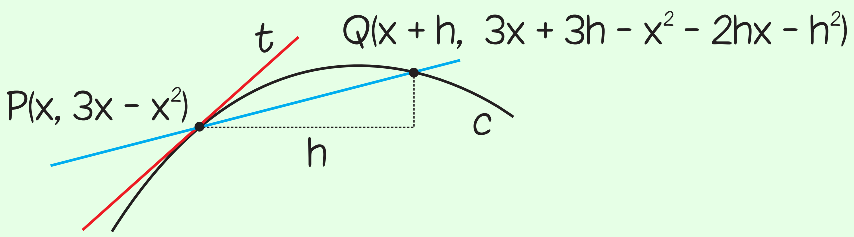
$$g(x) = \frac{x+5}{2}$$

$$\Rightarrow g(0) = \frac{(0)+5}{2} = \frac{5}{2}$$

$$\Rightarrow g(2) = \frac{(2)+5}{2} = \frac{7}{2}$$

$$\therefore g(0) + g(2) = \tfrac{5}{2} + \tfrac{7}{2} = \tfrac{12}{2} = 6$$

6 (b)



$$y = 3(x) - (x)^2$$
$$x = x: y = 3x - x^2$$

$$y = 3(x) - (x)^2$$
$$x = x + h: y = 3(x+h) - (x+h)^2$$
$$= 3(x+h) - (x+h)(x+h)$$
$$= 3(x+h) - (x^2 + hx + hx + h^2)$$
$$= 3(x+h) - (x^2 + 2hx + h^2)$$
$$= 3x + 3h - x^2 - 2hx - h^2$$

$$Q(\overset{x_2}{x+h},\ \overset{y_2}{3x+3h-x^2-2hx-h^2})$$
$$\downarrow \qquad \downarrow$$
$$P(\overset{x_1}{x},\ \overset{y_1}{3x-x^2})$$

$$\boxed{m = \frac{y_2 - y_1}{x_2 - x_1}}$$

$$\text{Slope of } PQ = \frac{(3x+3h-x^2-2hx-h^2)-(3x-x^2)}{x+h-x}$$
$$= \frac{3x+3h-x^2-2hx-h^2-3x+x^2}{x+h-x}$$
$$= \frac{3h-2hx-h^2}{h}$$
$$= \frac{h(3-2x-h)}{h} = 3-2x-h$$

$$\frac{dy}{dx} = \lim_{h\to 0}(3-2x-h) = 3-2x$$

6 (c) (i)

$$\boxed{y = x^n \Rightarrow \frac{dy}{dx} = nx^{n-1}}$$

Constant Rule: If y = Constant $\Rightarrow \frac{dy}{dx} = 0$

Multiply by a constant rule: If $y = cu$, where c is a constant and u is a function of x, $\frac{dy}{dx} = c \times \frac{du}{dx}$.

$$f(x) = x^2 + px + 10$$
$$\Rightarrow f'(x) = 2x + p\times 1 + 0 = 2x + p$$

6 (c) (ii)

To find the turning point which you are told is a minimum, put $f'(x) = 0$.

$$\boxed{\text{Turning Point} \Rightarrow \frac{dy}{dx} = 0}$$

$$f'(x) = 0 \Rightarrow 2x + p = 0$$

You are told that this minimum is at $x = 3$.

$$\therefore 2(3) + p = 0 \Rightarrow 6 + p = 0 \Rightarrow p = -6$$

6 (c) (iii)

Equation of a line: $\boxed{y - y_1 = m(x - x_1)}$

$$y = f(x) = x^2 - 6x + 10 \Rightarrow \frac{dy}{dx} = 2x - 6$$
$$\left(\frac{dy}{dx}\right)_{x=0} = 2(0) - 6 = -6 \Rightarrow m = -6$$

Point of contact is $(0, 10) = (x_1, y_1)$.

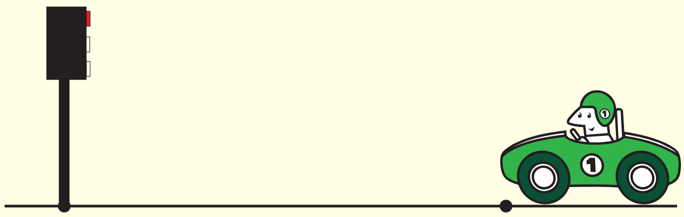
$$y - y_1 = m(x - x_1)$$
$$\Rightarrow y - 10 = -6(x - 0)$$
$$\Rightarrow y - 10 = -6x$$
$$\Rightarrow 6x + y - 10 = 0$$

7 (a) Differentiate $9+3x-5x^2$ with respect to x.

(b) (i) Differentiate $(3x^2-2)(x^2+4)$ with respect to x.

(ii) Given that $y=\frac{x^2}{x-1}$, find $\frac{dy}{dx}$ when $x = 3$.

(c) A car begins to slow down at p in order to stop at a red traffic light at q.



The distance of the car from p, after t seconds, is given by

$$s=12t-\tfrac{3}{2}t^2$$

where s is in metres.

(i) Find the speed of the car as it passes p.

(ii) Find the time taken to stop.

(iii) The car stops exactly at q. Find the distance from p to q.

SOLUTION

7 (a)

$$\boxed{y=x^n \Rightarrow \frac{dy}{dx}=nx^{n-1}}$$

CONSTANT RULE: If y = Constant $\Rightarrow \frac{dy}{dx}=0$

MULTIPLY BY A CONSTANT RULE: If $y = cu$, where c is a constant and u is a function of x, $\frac{dy}{dx}=c\times\frac{du}{dx}$.

$y=9+3x-5x^2$

$\Rightarrow \frac{dy}{dx}=0+3\times1-5\times2x$

$\Rightarrow \frac{dy}{dx}=3-10x$

7 (b) (i)

$y=(3x^2-2)(x^2+4)$

$u=(3x^2-2)\Rightarrow \frac{du}{dx}=3\times2x-0=6x$

$v=(x^2+4)\Rightarrow \frac{dv}{dx}=2x+0=2x$

$$\boxed{\frac{dy}{dx}=u\frac{dv}{dx}+v\frac{du}{dx}}$$

$\therefore \frac{dy}{dx}=u\frac{dv}{dx}+v\frac{dv}{dx}=(3x^2-2)(2x)+(x^2+4)(6x)$

$\Rightarrow \frac{dy}{dx}=6x^3-4x+6x^3+24x=12x^3+20x$

7 (b) (ii)

$$y=\frac{x^2}{x-1}$$

$$u=x^2 \Rightarrow \frac{du}{dx}=2x$$
$$v=x-1 \Rightarrow \frac{dv}{dx}=1$$

$$\frac{dy}{dx}=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$$

$$\frac{dy}{dx}=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}=\frac{(x-1)(2x)-x^2(1)}{(x-1)^2}$$
$$\Rightarrow \frac{dy}{dx}=\frac{2x^2-2x-x^2}{(x-1)^2}$$
$$\Rightarrow \frac{dy}{dx}=\frac{x^2-2x}{(x-1)^2}$$
$$\Rightarrow \left(\frac{dy}{dx}\right)_{x=3}=\frac{(3)^2-2(3)}{((3)-1)^2}=\frac{9-6}{(2)^2}$$
$$\Rightarrow \left(\frac{dy}{dx}\right)_{x=3}=\frac{3}{4}$$

7 (c) (i)

Draw up a s, v, a table as shown on the right.
You are asked to calculate the speed v at t = 0 seconds.

$$v=12-3t=12-3(0)=12 \text{ m/s}$$

$$v=\frac{ds}{dt}$$

$$a=\frac{dv}{dt}$$

7 (c) (ii)

When the car stops its speed v is zero. Put the speed equation equal to zero and solve for t.

$$v=12-3t \Rightarrow 0=12-3t$$
$$\Rightarrow 3t=12$$
$$\therefore t=4 \text{ s}$$

$$s=12t-\tfrac{3}{2}t^2$$
$$v=\frac{ds}{dt}=12-\tfrac{3}{2}\times 2t=12-3t$$
$$a=\frac{dv}{dt}=0-3=-3$$

7 (c) (iii)

It takes 4 s to stop. Put t = 4 in the distance s equation.

$$s=12t-\tfrac{3}{2}t^2$$
$$\Rightarrow s=12(4)-\tfrac{3}{2}(4)^2$$
$$\Rightarrow s=48-\tfrac{3}{2}(16)$$
$$\Rightarrow s=48-24$$
$$\therefore s=24 \text{ m}$$

8 Let $f(x)=\dfrac{1}{x-1}$, $x\in\mathbf{R}$, $x\neq 1$.

(i) Find $f(-3)$, $f(-1.5)$, $f(0.5)$, $f(1.5)$, $f(5)$.

(ii) Draw the graph of the function f from $x=-3$ to $x=5$.

(iii) On the same diagram, draw the graph of the function

$$g(x)=x+1$$

in the domain $-2\leq x\leq 2$, $x\in\mathbf{R}$.

(iv) Use your graphs to estimate the values of x for which $f(x)=g(x)$.

(v) Find, using algebra, the values of x for which $f(x)=g(x)$.

Solution

8 (i)

$$f(x)=\frac{1}{x-1}$$

$$f(-3)=\frac{1}{(-3)-1}=-\frac{1}{4}=-0.25$$

$$f(-1.5)=\frac{1}{(-1.5)-1}=-\frac{1}{2.5}=-0.4$$

$$f(0.5)=\frac{1}{(0.5)-1}=-\frac{1}{0.5}=-2$$

$$f(1.5)=\frac{1}{(1.5)-1}=\frac{1}{0.5}=2$$

$$f(5)=\frac{1}{(5)-1}=\frac{1}{4}=0.25$$

x	$f(x)$
–3	–0.25
–1.5	–0.4
0.5	–2
1.5	2
5	0.25

8 (ii)

Put $x-1=0 \Rightarrow x=1$ is the asymptote.

8 (iii)

$g(x) = x + 1$ is a straight line graph so you just need 2 points to draw the graph. Use the end values of the domain.

$x=-2: g(x)=x+1 \Rightarrow g(-2)=(-2)+1=-1 \Rightarrow (-2, -1)$ is a point.

$x=2: g(x)=x+1 \Rightarrow g(2)=(2)+1=3 \Rightarrow (2, 3)$ is a point.

Plot these two points using the same axes and draw a straight line through them.

8 (iv)

Find out where the two graphs intersect and read off the x values.

You can see that $x = 1.4$ and $x=-1.4$.

8 (v)

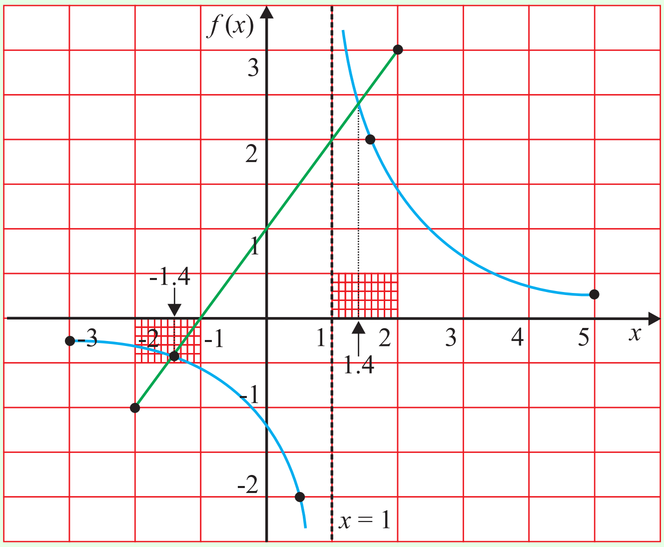
$f(x)=g(x) \Rightarrow \dfrac{1}{x-1}=x+1$ [Multiply across by $(x-1)$.]

$\Rightarrow 1=(x+1)(x-1)$ [Multiply out the brackets.]

$\Rightarrow 1=x^2-1$

$\Rightarrow 2=x^2$

$\Rightarrow x=\pm\sqrt{2}$



x	$f(x)$
–3	–0.25
–1.5	–0.4
0.5	–2
1.5	2
5	0.25