DIFFERENTIATION & FUNCTIONS (Q 6, 7 & 8, PAPER 1)

2004

6 (a) Let $g(x)=1-kx$.

Given that $g(-3)=13$, find the value of k.

(b) Let $f(x)=x^3-3x^2+1$, $x\in\mathbf{R}$.

(i) Find $f(-1)$ and $f(3)$.

(ii) Find $f'(x)$, the derivative of $f(x)$.

(iii) Find the co-ordinates of the local maximum point and of the local minimum point of the curve $y=f(x)$.

(iv) Draw the graph of the function f in the domain $-1\le x\le 3$.

Use your graph to:

(v) estimate the range of values of x for which $f(x)<0$ and $x>0$

(vi) estimate the range of values of x for which $f'(x)<0$.

SOLUTION

6 (a)

$g(x)=1-kx$

$g(-3)=13\Rightarrow 1-k(-3)=13$

$\Rightarrow 1+3k=13$

$\Rightarrow 3k=12$

$\therefore k=4$

6 (b) (i)

$f(x)=x^3-3x^2+1$

$\therefore\ f(-1)=(-1)^3-3(-1)^2+1=-1-3(1)+1=-1-3+1=-3$

$\therefore\ f(3)=(3)^3-3(3)^2+1=27-3(9)+1=27-27+1=1$

6 (b) (ii)

$$\boxed{y=x^n\Rightarrow\frac{dy}{dx}=nx^{n-1}}$$

CONSTANT RULE: If y = Constant $\Rightarrow\frac{dy}{dx}=0$

MULTIPLY BY A CONSTANT RULE: If $y=cu$, where c is a constant and u is a function of x, $\frac{dy}{dx}=c\times\frac{du}{dx}$.

$f(x)=x^3-3x^2+1$

$\Rightarrow\ f'(x)=3x^2-3\times 2x+0=3x^2-6x$

6 (b) (iii)

$y = f(x) = x^3 - 3x^2 + 1$

$\frac{dy}{dx} = f'(x) = 3x^2 - 6x$

$\frac{dy}{dx} = 0 \Rightarrow 3x^2 - 6x = 0$

$\Rightarrow 3x(x-2) = 0$

$\therefore x = 0,\ 2$

Turning Point: $\frac{dy}{dx} = 0$

$y = f(0) = (0)^3 - 3(0)^2 + 1 = 1 \Rightarrow (0,\ 1)$ is a local maximum.

$y = f(2) = (2)^3 - 3(2)^2 + 1 = 8 - 12 + 1 = -3 \Rightarrow (2,\ -3)$ is a local minimum.

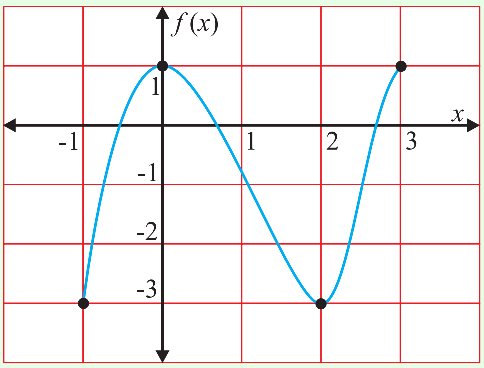
[(0, 1) is the local maximum as its *y* coordinate is greater than the *y* coordinate of the other point.]

6 (b) (iv)

You have enough points already to draw the cubic function.

You have from part **(i)** the starting and finishing points: $(-1,\ -3),\ (3,\ 1)$

From part **(iii)** you have the turning points: Local maximum (0, 1), Local minimum $(2,\ -3)$

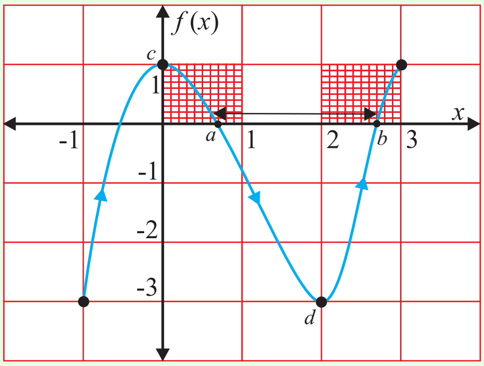


6 (b) (v)

$f(x) > 0$: Above the x-axis; $f(x) < 0$: Below the x-axis

What values of x is the graph below the x-axis ($f(x) < 0$) and to the right of the y-axis ($x > 0$)?
You can see from the graph that the values a and b satisfy this condition.

$\therefore 0.7 < x < 2.7$



6 (b) (vi)

$f'(x) < 0$: Curve is decreasing; $f'(x) > 0$: Curve is increasing.

You are being asked for what values of x is the curve decreasing. You can see from the graph it decreases as you go from left to right from c to d, i.e. values of x from 0 to 2.

$\therefore 0 < x < 2$

7 (a) Differentiate with respect to x:
(i) $2x^5$
(ii) $4(3-x^2)$.

(b) (i) Differentiate $(x^2-4)(x^2+3x)$ with respect to x.

(ii) Given that $y=(x^2-2x-3)^3$, show that $\frac{dy}{dx}=0$ when $x = 1$.

(c) A jet is moving along an airport runway. At the instant it passes a marker it begins to accelerate for take-off. From the time the jet passes the marker, its distance from the marker is given by

$$s = 2t^2 + 3t,$$

where s is in metres and t is in seconds.
(i) Find the speed of the jet at the instant it passes the marker ($t = 0$).
(ii) The jet has to reach a speed of 83 metres per second to take off. After how many seconds will the jet reach this speed?
(iii) How far is the jet from the marker at that time?
(iv) Find the acceleration of the jet.

SOLUTION

7 (a) (i)

$$y = x^n \Rightarrow \frac{dy}{dx} = nx^{n-1}$$

CONSTANT RULE: If y = Constant $\Rightarrow \frac{dy}{dx} = 0$

MULTIPLY BY A CONSTANT RULE: If $y = cu$, where c is a constant and u is a function of x, $\frac{dy}{dx} = c \times \frac{du}{dx}$.

$$y = 2x^5 \Rightarrow \frac{dy}{dx} = 2\times 5x^4 = 10x^4$$

7 (a) (ii)

$$y = 4(3 - x^2) = 12 - 4x^2$$
$$\Rightarrow \frac{dy}{dx} = 0 - 4 \times 2x = -8x$$

7 (b) (i)

$$y = (x^2 - 4)(x^2 + 3x)$$

$$u = (x^2 - 4) \Rightarrow \frac{du}{dx} = 2x - 0 = 2x$$
$$v = (x^2 + 3x) \Rightarrow \frac{dv}{dx} = 2x + 3$$

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$\therefore \frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx} = (x^2 - 4)(2x + 3) + (x^2 + 3x)(2x)$$
$$\Rightarrow \frac{dy}{dx} = 2x^3 + 3x^2 - 8x - 12 + 2x^3 + 6x^2$$
$$\Rightarrow \frac{dy}{dx} = 4x^3 + 9x^2 - 8x - 12$$

7 (b) (ii)

$$y = (x^2 - 2x - 3)^3$$

Move the power down in front of the bracket.
Take one away from the power.
Multiply by the differentiation of the inside of the bracket.

$$u = (x^2 - 2x - 3) \Rightarrow \frac{du}{dx} = 2x - 2$$

$$y = (x^2 - 2x - 3)^3$$
$$\Rightarrow \frac{dy}{dx} = 3(x^2 - 2x - 3)^2(2x - 2)$$
$$\Rightarrow \frac{dy}{dx} = (6x - 6)(x^2 - 2x - 3)^2$$

7 (c) (i)

Draw up a *s*, *v*, *a* table as shown on the right.
You are asked to find the speed *v* at time *t* = 0.

$$v = 4t + 3 = 4(0) + 3 = 3 \text{ m/s}$$

$$s = 2t^2 + 3t$$
$$v = \frac{ds}{dt} = 4t + 3$$
$$a = \frac{dv}{dt} = 4$$

$$v = \frac{ds}{dt}$$
$$a = \frac{dv}{dt}$$

7 (c) (ii)
You are asked to find the time t it takes to reach a speed v of 83 metres per second.

$v = 4t + 3 \Rightarrow 83 = 4t + 3$

$\Rightarrow 4t = 80 \Rightarrow t = 20 \text{ s}$

7 (c) (iii)
You are asked to find the distance s travelled after a time t of 20 s.

$s = 2t^2 + 3t \Rightarrow s = 2(20)^2 + 3(20)$

$\Rightarrow s = 800 + 60 = 860 \text{ m}$

7 (c) (iv)
$a = 4 \text{ m/s}^2$

8 (a) Let $g(x) = 3x - 7$.

(i) Find $g(7)$.

(ii) Find the value of k for which $g(7) = k[g(0)]$.

(b) Differentiate $x^2 + 3x$ with respect to x from first principles.

(c) Let $f(x) = \dfrac{1}{x+3}$, $x \in \mathbf{R}$, $x \neq -3$.

(i) Find $f'(x)$, the derivative of $f(x)$.

(ii) There are two points on the curve $y = f(x)$ at which the slope of the tangent is -1. Find the co-ordinates of these two points.

(iii) Show that no tangent to the curve $y = f(x)$ has a slope of 1.

SOLUTION

8 (a) (i)

$g(x) = 3x - 7$

$\Rightarrow g(7) = 3(7) - 7 = 21 - 7 = 14$

8 (a) (ii)

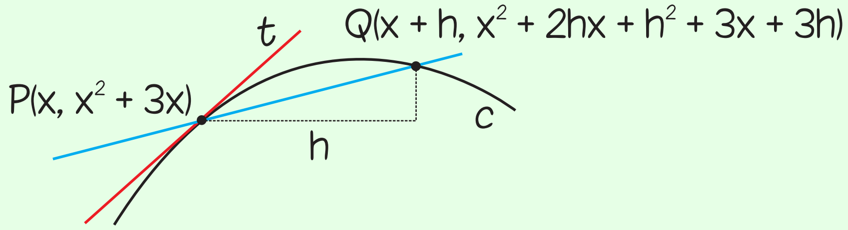
$g(7) = k[g(0)]$

$\Rightarrow 14 = k[3(0) - 7]$

$\Rightarrow 14 = k[-7]$

$\therefore k = -2$

8 (b)



$$y=(x)^2+3(x)$$
$$x=x: y=x^2+3x$$

$$y=(x)^2+3(x)$$
$$\begin{aligned} x=x+h: y &= (x+h)^2+3(x+h) \\ &= (x+h)(x+h)+3(x+h) \\ &= (x^2+hx+hx+h^2)+3(x+h) \\ &= (x^2+2hx+h^2)+3(x+h) \\ &= x^2+2hx+h^2+3x+3h \end{aligned}$$

$$Q(\overset{x_2}{x+h},\ \overset{y_2}{x^2+2hx+h^2+3x+3h})$$
$$P(\underset{}{\overset{x_1}{x}},\ \overset{y_1}{x^2+3x})$$

$$m=\frac{y_2-y_1}{x_2-x_1}$$

$$\begin{aligned} \text{Slope of } PQ &= \frac{(x^2+2hx+h^2+3x+3h)-(x^2+3x)}{x+h-x} \\ &= \frac{x^2+2hx+h^2+3x+3h-x^2-3x}{x+h-x} \\ &= \frac{2hx+h^2+3h}{h} \\ &= \frac{h(2x+h+3)}{h}=2x+h+3 \end{aligned}$$

$$\frac{dy}{dx}=\lim_{h\to 0}(2x+h+3)=2x+3$$

8 (c) (i)

Move the power down in front of the bracket.
Take one away from the power.
Multiply by the differentiation of the inside of the bracket.

$$f(x)=\frac{1}{x+3}=(x+3)^{-1}$$

$$\Rightarrow f'(x)=-1(x+3)^{-2}(1)=-\frac{1}{(x+3)^2}$$

8 (c) (ii)

Steps: Finding points of contact given the slope, m, of the tangent

1. Find $\dfrac{dy}{dx}$.

2. Put $\dfrac{dy}{dx}=m$ and solve for x.

3. Find the corresponding y values.

1. $f'(x)=\dfrac{dy}{dx}=-\dfrac{1}{(x+3)^2}$

2. $-\dfrac{1}{(x+3)^2}=-1 \Rightarrow 1=(x+3)^2$

$\Rightarrow x+3=\pm 1$

$\therefore x=-2,\ -4$

3. $x=-2: y=f(-2)=\dfrac{1}{(-2)+3}=\dfrac{1}{1}=1 \Rightarrow (-2,\ 1)$ is a points of contact with the tangent.

$x=-4: y=f(-4)=\dfrac{1}{(-4)+3}=\dfrac{1}{-1}=-1 \Rightarrow (-4,\ -1)$ is a points of contact with the tangent.

8 (c) (iii)

Put $\dfrac{dy}{dx}=1$ and show that there exists no solutions for x.

$\dfrac{dy}{dx}=1 \Rightarrow -\dfrac{1}{(x+3)^2}=1$

$\Rightarrow -1=(x+3)^2$

$\Rightarrow \sqrt{-1}=(x+3)$

$\sqrt{-1}$ has no real solutions. Therefore, no tangent to the curve has a slope of 1.