DIFFERENTIATION & FUNCTIONS (Q 6, 7 & 8, PAPER 1)

2003

6 (a) Let $g(x)=\frac{2x}{3}-1.$

Find the value of x for which $g(x) = 5$.

(b) Differentiate x^2-2x with respect to x from first principles.

(c) Let $f(x)=3-5x-2x^2,\ x\in\mathbf{R}.$

(i) Find $f'(x)$, the derivative of $f(x)$, and hence find the co-ordinates of the local maximum point of the curve $y = f(x)$.

(ii) Solve the equation $f(x) = 0$.

(iii) Use your answers from parts (i) and (ii) to sketch the graph of

$f:x\rightarrow 3-5x-2x^2,$ showing scaled and labelled axes.

SOLUTION

6 (a)

$g(x)=5$

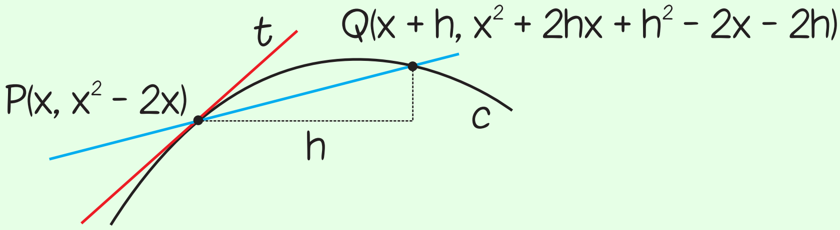
$\Rightarrow \frac{2x}{3}-1=5$

$\Rightarrow \frac{2x}{3}=6$ [Multiply across by 3.]

$\Rightarrow 2x=18$

$\Rightarrow x=9$

6 (b)



$$y=(x)^2-2(x)$$
$$x=x: y=x^2-2x$$

$$y=(x)^2-2(x)$$
$$\begin{aligned} x=x+h: y&=(x+h)^2-2(x+h) \\ &=(x+h)(x+h)-2(x+h) \\ &=(x^2+hx+hx+h^2)-2(x+h) \\ &=(x^2+2hx+h^2)-2(x+h) \\ &=x^2+2hx+h^2-2x-2h \end{aligned}$$

$$Q(\overset{x_2}{x+h},\ \overset{y_2}{x^2+2hx+h^2-2x-2h})$$
$$\downarrow \qquad \downarrow$$
$$P(\overset{x_1}{x},\ \overset{y_1}{x^2-2x})$$

$$m=\frac{y_2-y_1}{x_2-x_1}$$

$$\begin{aligned} \text{Slope of } PQ &=\frac{(x^2+2hx+h^2-2x-2h)-(x^2-2x)}{x+h-x} \\ &=\frac{x^2+2hx+h^2-2x-2h-x^2+2x}{x+h-x} \\ &=\frac{2hx+h^2-2h}{h} \\ &=\frac{h(2x+h-2)}{h}=2x+h-2 \end{aligned}$$

$$\frac{dy}{dx}=\lim_{h\to 0}(2x+h-2)=2x-2$$

6 (c) (i)

$y = f(x) = 3 - 5x - 2x^2$

$\dfrac{dy}{dx} = f'(x) = -5 - 4x$

$\dfrac{d^2y}{dx^2} = f''(x) = -4$

$\dfrac{dy}{dx} = 0 \Rightarrow -5 - 4x = 0$

$\Rightarrow -5 = 4x$

$\therefore x = -\frac{5}{4}$

You are told that this point is a maximum.

$y = f(-\frac{5}{4}) = 3 - 5(-\frac{5}{4}) - 2(-\frac{5}{4})^2$

$= 3 + \frac{25}{4} - 2(\frac{25}{16}) = 3 + \frac{25}{4} - \frac{25}{8} = \frac{49}{8} \Rightarrow (-\frac{5}{4}, \frac{49}{8})$ is the local maximum.

6 (c) (ii)

$f(x) = 0 \Rightarrow 3 - 5x - 2x^2 = 0$

$\Rightarrow 2x^2 + 5x - 3 = 0$

$\Rightarrow (2x - 1)(x + 3) = 0$

$\Rightarrow x = -3, \frac{1}{2}$

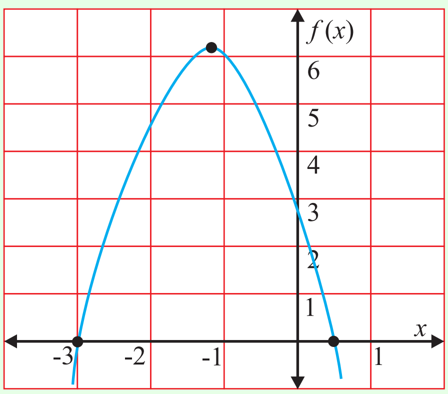
Graph cuts x-axis: Put $f(x) = 0$.

6 (c) (iii)

From part **(i)** you have found out the coordinates of the local maximum: $(-\frac{5}{4}, \frac{49}{8})$

From part **(ii)** you have found out where the graph cuts the x-axis: $(-3, 0)$, $(\frac{1}{2}, 0)$

Draw the quadratic graph using these points.



7 (a) Differentiate with respect to x:

(i) x^3

(ii) $\dfrac{x^2 - x^4}{2}$.

(b) (i) Differentiate $(3x^3 - 2x^2 + 2)^4$ with respect to x.

(ii) Given that $y = (5x^2 + 3)(4 - x^2)$, find $\dfrac{dy}{dx}$ when $x = 1$.

(c) A train is travelling along a track. Suddenly, the brakes are applied. From the time the brakes are applied ($t = 0$ seconds), the distance travelled by the train, in metres, is given by

$$s = 30t - \tfrac{1}{4}t^2.$$

(i) What is speed of the train at the moment the brakes are applied?
(ii) How many seconds does it take for the train to come to rest?
(iii) How far does the train travel in that time?

SOLUTION

7 (a) (i)

$$\boxed{y = x^n \Rightarrow \frac{dy}{dx} = nx^{n-1}}$$

CONSTANT RULE: If y = Constant $\Rightarrow \dfrac{dy}{dx} = 0$

MULTIPLY BY A CONSTANT RULE: If $y = cu$, where c is a constant and u is a function of x, $\dfrac{dy}{dx} = c \times \dfrac{du}{dx}$.

$$y = x^3 \Rightarrow \frac{dy}{dx} = 3x^2$$

7 (a) (ii)

$$y = \frac{x^2 - x^4}{2} = \tfrac{1}{2}x^2 - \tfrac{1}{2}x^4$$

$$\Rightarrow \frac{dy}{dx} = \tfrac{1}{2} \times 2x - \tfrac{1}{2} \times 4x^3 = x - 2x^3$$

7 (b) (i)

Move the power down in front of the bracket.
Take one away from the power.
Multiply by the differentiation of the inside of the bracket.

$$\boxed{u = (3x^3 - 2x^2 + 2) \Rightarrow \frac{du}{dx} = 9x^2 - 4x}$$

$$y = (3x^3 - 2x^2 + 2)^4 \Rightarrow \frac{dy}{dx} = 4(3x^3 - 2x^2 + 2)^3(9x^2 - 4x) = (36x^2 - 16x)(3x^3 - 2x^2 + 2)^3$$

7 (b) (ii)

$$\boxed{\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}}$$

$$u = (5x^2 + 3) \Rightarrow \frac{du}{dx} = 10x$$

$$v = (4 - x^2) \Rightarrow \frac{dv}{dx} = -2x$$

$$y = (5x^2 + 3)(4 - x^2)$$

$$\Rightarrow \frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx} = (5x^2 + 3)(-2x) + (4 - x^2)(10x)$$

$$\Rightarrow \frac{dy}{dx} = -10x^3 - 6x + 40x - 10x^3$$

$$\Rightarrow \frac{dy}{dx} = -20x^3 + 34x$$

$$\therefore \left(\frac{dy}{dx}\right)_{x=1} = -20(1)^3 + 34(1) = -20 + 34 = 14$$

7 (c) (i)

Draw up an *s*, *v*, *a* table as shown on the right.
You are asked to find the speed *v* at time *t* = 0.

$$\boxed{v = \frac{ds}{dt}}$$

$$\boxed{a = \frac{dv}{dt}}$$

$$s = 30t - \tfrac{1}{4}t^2$$

$$v = \frac{ds}{dt} = 30 - \tfrac{1}{2}t$$

$$a = \frac{dv}{dt} = -\tfrac{1}{2}$$

$$v = 30 - \tfrac{1}{2}t \Rightarrow v = 30 - \tfrac{1}{2}(0) = 30 \text{ m/s}$$

7 (c) (ii)

You are asked to find the time *t* it takes for the train to stop, i.e. *v* = 0 m/s.

$$v = 30 - \tfrac{1}{2}t \Rightarrow 0 = 30 - \tfrac{1}{2}t$$

$$\Rightarrow \tfrac{1}{2}t = 30 \Rightarrow t = 60 \text{ s}$$

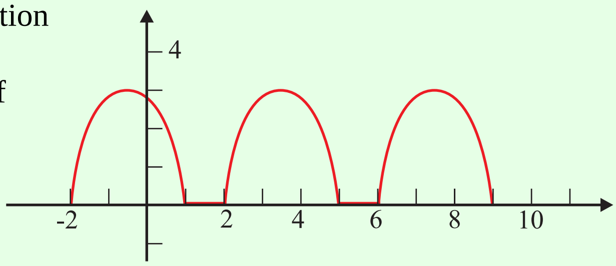
7 (c) (iii)

You are asked to find the distance s travelled after 60 s.

$$s = 30t - \tfrac{1}{4}t^2 \Rightarrow s = 30(60) - \tfrac{1}{4}(60)^2$$

$$= 1800 - \tfrac{1}{4}(3600) = 1800 - 900 = 900 \text{ m/s}$$

8 (a) Part of the graph of a periodic function is shown.
Write down the period and range of the function.



(b) (i) The function g is defined for natural numbers by the rule:

$$g(x) = \begin{cases} 0 \text{ if is even.} \\ 1 \text{ if is odd} \end{cases}$$

Find $g(13) + g(14) + g(15)$.

(ii) Given that $h(x) = x^2$, write down $h(x + 3)$.
Hence, find the value of x for which $h(x) = h(x + 3)$.

(c) Let $f(x) = x^3 + 2x^2 - 1$.

(i) Find $f'(x)$, the derivative of $f(x)$.

(ii) L is the tangent to the curve $y = f(x)$ at $x = -\frac{2}{3}$.
Find the slope of L.

(iii) Find the two values of x at which the tangents to the curve $y = f(x)$ are perpendicular to L.

Solution

8 (a)

Period: Length of the wave along the x-axis before it repeats itself.

Range: The interval between the lowest y value and the highest y value.

Period = 4
Range = [0, 3]

8 (b) (i)

$g(13) = 1$ (because 13 is an odd number)
$g(14) = 0$ (because 14 is an even number)
$g(15) = 1$ (because 15 is an odd number)

$\therefore g(13) + g(14) + g(15) = 0 + 1 + 0 = 1$

8 (b) (ii)

$h(x) = x^2$

$\therefore h(x+3) = (x+3)^2 = x^2 + 6x + 9$

$h(x) = h(x+3)$

$\therefore x^2 = x^2 + 6x + 9 \Rightarrow 0 = 6x + 9$

$\Rightarrow -9 = 6x \Rightarrow -\frac{9}{6} = x$

$\therefore x = -\frac{3}{2}$

8 (c) (i)

$$\boxed{y = x^n \Rightarrow \frac{dy}{dx} = nx^{n-1}}$$

Constant Rule: If y = Constant $\Rightarrow \frac{dy}{dx} = 0$

Multiply by a constant rule: If $y = cu$, where c is a constant and u is a function of x, $\frac{dy}{dx} = c \times \frac{du}{dx}$.

$$f(x) = x^3 + 2x^2 - 1$$
$$\therefore f'(x) = 3x^2 + 2 \times 2x - 0 = 3x^2 + 4x$$

8 (c) (ii)

You are being asked to find the slope $\frac{dy}{dx}$ at $x = -\frac{2}{3}$.

$$\left(\frac{dy}{dx}\right)_{x=-\frac{2}{3}} = 3(-\tfrac{2}{3})^2 + 4(-\tfrac{2}{3}) = 3(\tfrac{4}{9}) - \tfrac{8}{3} = \tfrac{4}{3} - \tfrac{8}{3} = -\tfrac{4}{3}$$

8 (c) (iii)

Steps: Finding points of contact given the slope, m, of the tangent

1. Find $\frac{dy}{dx}$.
2. Put $\frac{dy}{dx} = m$ and solve for x.
3. Find the corresponding y values.

L has a slope of $-\frac{4}{3}$. The perpendicular slope is $\frac{3}{4}$.

Finding the perpendicular slope: Invert the slope and change its sign.

1. $\frac{dy}{dx} = 3x^2 + 4x$

2. $\frac{dy}{dx} = \frac{3}{4} \Rightarrow 3x^2 + 4x = \frac{3}{4}$ [Multiply across by 4.]

$\Rightarrow 12x^2 + 16x = 3$

$\Rightarrow 12x^2 + 16x - 3 = 0$

$\Rightarrow (6x - 1)(2x + 3) = 0$

$\therefore x = -\frac{3}{2}, \frac{1}{6}$

Step **3** is not needed as you are asked to find the y values only.