DIFFERENTIATION & FUNCTIONS (Q 6, 7 & 8, PAPER 1)

2002

6 (a) Let $f(x)=\frac{1}{3}(x-8)$ for $x \in \mathbf{R}$.
Evaluate $f(5)$.

(b) (i) Find $\frac{dy}{dx}$ where $y=(x-1)^7$ and evaluate your answer at $x = 2$.

(ii) Find $\frac{dy}{dx}$ where $y=(x^3-3)(x^2-4)$ and simplify your answer.

(c) Let $f(x)=x^3-ax+7$ for all $x \in \mathbf{R}$ and for $a \in \mathbf{R}$.

(i) The slope of the tangent to the curve $y = f(x)$ at $x = 1$ is -9.
Find the value of a.

(ii) Hence, find the co-ordinates of the local maximum point and the local minimum point on the curve $y = f(x)$.

SOLUTION

6 (a)

$$f(x)=\tfrac{1}{3}(x-8)$$

$$\Rightarrow f(5)=\tfrac{1}{3}(5-8)=\tfrac{1}{3}(-3)=-1$$

6 (b) (i)

Move the power down in front of the bracket.
Take one away from the power.
Multiply by the differentiation of the inside of the bracket.

$$y=(x-1)^7$$

$$\Rightarrow \frac{dy}{dx}=7(x-1)^6(1)=7(x-1)^6$$

$$u=(x-1) \Rightarrow \frac{du}{dx}=1$$

6 (b) (ii)

$$\boxed{\frac{dy}{dx}=u\frac{dv}{dx}+v\frac{du}{dx}}$$

$$y=(x^3-3)(x^2-4)$$

$$\Rightarrow \frac{dy}{dx}=u\frac{dv}{dx}+v\frac{du}{dx}=(x^3-3)(2x)+(x^2-4)(3x^2)$$

$$\Rightarrow \frac{dy}{dx}=2x^4-6x+3x^4-12x^2=5x^4-12x^2-6x$$

$$u=(x^3-3) \Rightarrow \frac{du}{dx}=3x^2$$

$$v=(x^2-4) \Rightarrow \frac{dv}{dx}=2x$$

6 (c) (i)

Find the slope of the curve $\frac{dy}{dx}$ at $x = 1$, $\left(\frac{dy}{dx}\right)_{x=1}$, and put it equal to -9.

$$y = f(x) = x^3 - ax + 7$$

$$\therefore \frac{dy}{dx} = 3x^2 - a$$

$$\Rightarrow \left(\frac{dy}{dx}\right)_{x=1} = 3(1)^2 - a = -9$$

$$\Rightarrow 3 - a = -9$$

$$\therefore a = 12$$

6 (c) (ii)

$$y = f(x) = x^3 - 12x + 7$$

$$\frac{dy}{dx} = 3x^2 - 12$$

$$\frac{d^2y}{dx^2} = 6x$$

$$\frac{dy}{dx} = 0 \Rightarrow 3x^2 - 12 = 0$$

$$\boxed{\text{Turning Point: } \frac{dy}{dx} = 0}$$

$$\Rightarrow 3(x^2 - 4) = 0$$

$$\Rightarrow 3(x+2)(x-2) = 0$$

$$\therefore x = -2,\ 2$$

$x = -2: y = f(-2) = (-2)^3 - 12(-2) + 7 = -8 + 24 + 7 = 23 \Rightarrow (-2,\ 23)$ is a local maximum.

$x = 2: y = f(2) = (2)^3 - 12(2) + 7 = 8 - 24 + 7 = 23 \Rightarrow (2,\ -9)$ is a local minimum.

[(–2, 23) is the local maximum as its *y* coordinate is greater than the *y* coordinate of the other point.]

7 (a) Differentiate $7x^3 - 3x^2 + 9x$ with respect to x.

(b) (i) Differentiate $x^5 - 17 + \dfrac{1}{x^5}$ with respect to x.

(ii) Differentiate $\dfrac{2x}{x-1}$ with respect to x and simplify your answer.

(c) A marble rolls along the top of a table. It starts to move at $t = 0$ seconds. The distance that it has travelled at t seconds is given by

$$s = 14t - t^2$$

where s is in centimetres.

(i) What distance has the marble travelled when $t = 2$ seconds?
(ii) What is the speed of the marble when $t = 5$ seconds?
(iii) When is the speed of the marble equal to zero?
(iv) What is the acceleration of the marble?

Solution

7 (a)

$$\boxed{y = x^n \Rightarrow \frac{dy}{dx} = nx^{n-1}}$$

Constant Rule: If y = Constant $\Rightarrow \dfrac{dy}{dx} = 0$

Multiply by a constant rule: If $y = cu$, where c is a constant and u is a function of x, $\dfrac{dy}{dx} = c \times \dfrac{du}{dx}$.

$$y = 7x^3 - 3x^2 + 9x$$

$$\Rightarrow \frac{dy}{dx} = 7 \times 3x^2 - 3 \times 2x + 9 = 21x^2 - 6x + 9$$

7 (b) (i)

$$y = x^5 - 17 + \frac{1}{x^5} = x^5 - 17 + x^{-5}$$

$$\Rightarrow \frac{dy}{dx} = 5x^4 - 0 - 5x^{-6}$$

$$\Rightarrow \frac{dy}{dx} = 5x^4 - \frac{5}{x^6}$$

7 (b) (ii)

$$y = \frac{2x}{x-1} \Rightarrow \frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2} = \frac{(x-1)2 - 2x(1)}{(x-1)^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{2x - 2 - 2x}{(x-1)^2} = -\frac{2}{(x-1)^2}$$

$$\boxed{\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}}$$

$$u = 2x \Rightarrow \frac{du}{dx} = 2$$

$$v = (x-1) \Rightarrow \frac{dv}{dx} = 1$$

7 (c) (i)

Draw up a *s*, *v*, *a* table as shown on the right.

you are asked to find the distance *s* travelled after a time t = 2 seconds.

$s = 14t - t^2 \Rightarrow s = 14(2) - (2)^2 = 28 - 4$

$\therefore s = 24$ cm

$$v = \frac{ds}{dt}$$

$$a = \frac{dv}{dt}$$

$$s = 14t - t^2$$

$$v = \frac{ds}{dt} = 14 - 2t$$

$$a = \frac{dv}{dt} = -2$$

7 (c) (ii)

You are asked to find the speed *v* when t = 5 seconds.

$v = 14 - 2t \Rightarrow v = 14 - 2(5) = 14 - 10$

$\therefore v = 4$ cm/s

7 (c) (iii)

You are asked to find the time *t* when the speed *v* is zero.

$v = 14 - 2t \Rightarrow 0 = 14 - 2t$

$\Rightarrow 2t = 14$

$\therefore t = 7$ s

7 (c) (iv)

$a = -2$ cm/s^2

8 Let $f(x) = \dfrac{1}{x+2}$.

(i) Find $f(-6)$, $f(-3)$, $f(-1)$, $f(0)$ and $f(2)$.

(ii) For what real value of *x* is $f(x)$ not defined?

(iii) Draw the graph of $f(x) = \dfrac{1}{x+2}$ for $-6 \le x \le 2$.

(iv) Find $f'(x)$, the derivative of $f(x)$.

(v) Find the two values of *x* at which the slope of the tangent to the graph is $-\frac{1}{9}$.

(vi) Show that there is no tangent to the graph of *f* that is parallel to the *x*-axis.

SOLUTION

8 (i)

$$f(x)=\frac{1}{x+2}$$

$$f(-6)=\frac{1}{-6+2}=-\frac{1}{4}=-0.25$$

$$f(-3)=\frac{1}{-3+2}=-\frac{1}{1}=-1$$

$$f(-1)=\frac{1}{-1+2}=\frac{1}{1}=1$$

$$f(0)=\frac{1}{0+2}=\frac{1}{2}=0.5$$

$$f(2)=\frac{1}{2+2}=\frac{1}{4}=0.25$$

8 (ii)

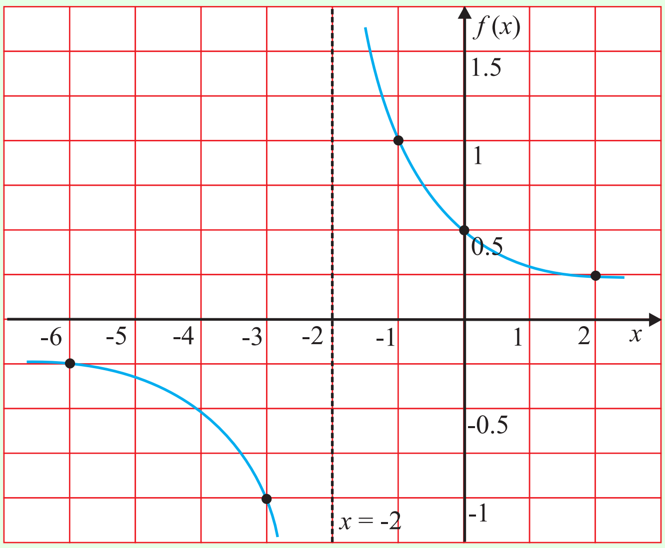
Put the bottom equal to zero. This will give the equation of the gap or the asymptote in the graph. It is also the value of x for which $f(x)$ is not defined.

$x+2=0 \Rightarrow x=-2$

8 (iii)

Use the values from part **(i)** and the asymptote equation in part **(ii)** to draw the graph.

Points: $(-6,\ 0.25)$, $(-3,\ -1)$, $(-1,\ 1)$, $(0,\ 0.5)$, $(2,\ 0.25)$



8 (iv)

$$f(x)=\frac{1}{(x+2)}=(x+2)^{-1}$$

$$\Rightarrow f'(x)=-1(x+2)^{-2}(1)=-\frac{1}{(x+2)^2}$$

8 (v)

STEPS: Finding points of contact given the slope, *m*, of the tangent

1. Find $\frac{dy}{dx}$.

2. Put $\frac{dy}{dx} = m$ and solve for *x*.

3. Find the corresponding *y* values.

1. $f'(x) = \frac{dy}{dx} = -\frac{1}{(x+2)^2}$

2. $-\frac{1}{(x+2)^2} = -\frac{1}{9} \Rightarrow 9 = (x+2)^2$ [Multiply across by $9(x + 2)^2$.]

$\Rightarrow \pm 3 = x + 2$

$\therefore x = -5, 1$

Step **3** is not required as you need to find the *x* values only.

8 (vi)

Any line parallel to the *x*-axis has a slope of zero.

Put $\frac{dy}{dx}$ equal to zero and show it has no solutions.

$\frac{dy}{dx} = 0 \Rightarrow -\frac{1}{(x+2)^2} = 0$ [Multiply across by $(x + 2)^2$.]

$\Rightarrow -1 = 0$

This equation is nonsense. Therefore, there are no tangents to the graph that are parallel to the *x*-axis.