Differentiation & Functions (Q 6, 7 & 8, Paper 1)

2001

6 (a) Let $g(x)=\dfrac{1}{x^2+1}$ for $x \in \mathbf{R}$.

Evaluate

(i) $g(2)$

(ii) $g(3)$ and write your answers as decimals.

(b) Let $f(x)=2-9x+6x^2-x^3$ for $x \in \mathbf{R}$.

(i) Find $f(-1)$, $f(2)$ and $f(5)$.

(ii) Find $f'(x)$, the derivative of $f(x)$.

(iii) Find the co-ordinates of the local maximum and the local minimum of $f(x)$.

(iv) Draw the graph of $f(x)$ in the domain $-1 \le x \le 5$.

(v) Use your graph to find the range of real values of k for which $f(x) = k$ has more than one solution.

Solution

6 (a)

$$g(x)=\frac{1}{x^2+1}$$

$$\Rightarrow g(2)=\frac{1}{(2)^2+1}=\frac{1}{4+1}=\frac{1}{5}=0.2$$

$$\Rightarrow g(3)=\frac{1}{(3)^2+1}=\frac{1}{9+1}=\frac{1}{10}=0.1$$

6 (b) (i)

$$f(x)=2-9x+6x^2-x^3$$

$$\Rightarrow f(-1)=2-9(-1)+6(-1)^2-(-1)^3=2+9+6(1)-(-1)$$

$$=2+9+6+1=18$$

$$\Rightarrow f(2)=2-9(2)+6(2)^2-(2)^3=2-18+6(4)-(8)$$

$$=2-18+24-8=0$$

$$\Rightarrow f(5)=2-9(5)+6(5)^2-(5)^3=2-45+6(25)-(125)$$

$$=2-45+150-125=-13$$

6 (b) (ii)

$$\boxed{y=x^n \Rightarrow \frac{dy}{dx}=nx^{n-1}}$$

Constant Rule: If y = Constant $\Rightarrow \dfrac{dy}{dx}=0$

Multiply by a constant rule: If $y = cu$, where c is a constant and u is a function of x, $\dfrac{dy}{dx}=c\times\dfrac{du}{dx}$.

$$f(x)=2-9x+6x^2-x^3$$

$$\Rightarrow f'(x)=0-9+6\times 2x-3x^2=-9+12x-3x^2$$

6 (b) (iii)

$$y = f(x) = 2 - 9x + 6x^2 - x^3$$

$$\frac{dy}{dx} = f'(x) = -9 + 12x - 3x^2$$

$$\frac{d^2y}{dx^2} = f''(x) = 12 - 6x$$

$$\frac{dy}{dx} = 0 \Rightarrow -9 + 12x - 3x^2 = 0$$

$$\boxed{\text{Turning Point: } \frac{dy}{dx} = 0}$$

$$\Rightarrow x^2 - 4x + 3 = 0$$

$$\Rightarrow (x-1)(x-3) = 0$$

$$\therefore x = 1,\ 3$$

$x = 1: y = f(1) = 2 - 9(1) + 6(1)^2 - (1)^3 = 2 - 9 + 6 - 1 = -2 \Rightarrow (1, -2)$ is a local minimum.

$x = 3: y = f(3) = 2 - 9(3) + 6(3)^2 - (3)^3 = 2 - 27 + 54 - 27 = 2 \Rightarrow (3, 2)$ is a local maximum.

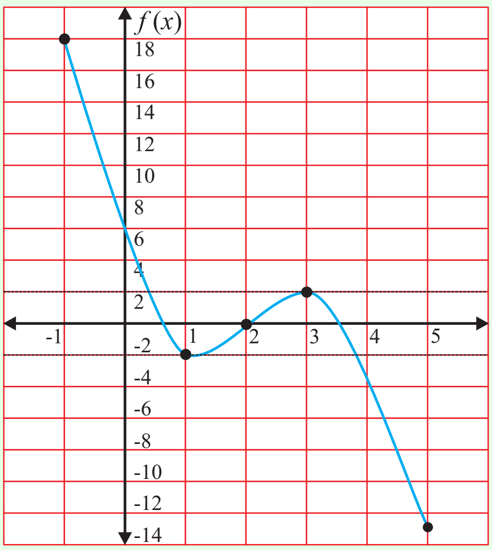
[(3, 2) is the local maximum as its y coordinate is greater than the y coordinate of the other point.]

6 (b) (iv)

You have sufficient information from parts **(i)** and **(iii)** to draw the cubic graph.

Points: $(-1, 18), (2, 0), (5, -13)$

Local maximum: $(3, 2)$; Local minimum: $(1, -2)$



6 (b) (v)

The line $f(x) = k$ is a horizontal line. You can see from the graph that all lines drawn between 2 and -2 will cut the graph in more than one place.

$\therefore -2 \le k \le 2$

7 (a) Differentiate with respect to x

(i) $6x^5 + x^2$

(ii) $(x-3)(x+3)$

(b) (i) Find $\frac{dy}{dx}$ when $y = \frac{x^2}{x-4}$, $x \neq 4$.

(ii) Find the value of $\frac{dy}{dx}$ at $x = 0$ when $y = (x^2 - 7x + 1)^5$.

(c) Two fireworks were fired straight up in the air at $t = 0$ seconds.
The height, h metres, which each firework reached above the ground t seconds after it was fired is given by

$$h = 80t - 5t^2.$$

The first firework exploded 5 seconds after it was fired.

(i) At what height was the first firework when it exploded?

(ii) At what speed was the first firework travelling when it exploded?

The second firework failed to explode and it fell back to the ground.

(iii) After how many seconds did the second firework reach its maximum height?

SOLUTION

7 (a) (i)

$$\boxed{y = x^n \Rightarrow \frac{dy}{dx} = nx^{n-1}}$$

CONSTANT RULE: If y = Constant $\Rightarrow \frac{dy}{dx} = 0$

MULTIPLY BY A CONSTANT RULE: If $y = cu$, where c is a constant and u is a function of x, $\frac{dy}{dx} = c \times \frac{du}{dx}$.

$$y = 6x^5 + x^2$$

$$\Rightarrow \frac{dy}{dx} = 6 \times 5x^4 + 2x = 30x^4 + 2x$$

7 (a) (ii)

You could use the product rule but it is easier to multiply out the brackets and differentiate term by term.

$$y = (x-3)(x+3) = x^2 + 3x - 3x - 9 = x^2 - 9$$

$$\Rightarrow \frac{dy}{dx} = 2x - 0 = 2x$$

7 (b) (i)

$$\boxed{\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}}$$

$$y = \frac{x^2}{(x-4)}$$

$$\boxed{\begin{aligned} u = x^2 &\Rightarrow \frac{du}{dx} = 2x \\ v = (x-4) &\Rightarrow \frac{du}{dx} = 1 \end{aligned}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2} = \frac{(x-4)(2x) - x^2(1)}{(x-1)^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{2x^2 - 8x - x^2}{(x-1)^2} = \frac{x^2 - 8x}{(x-1)^2}$$

7 (b) (ii)

Move the power down in front of the bracket.
Take one away from the power.
Multiply by the differentiation of the inside of the bracket.

$$u = x^2 - 7x + 1 \Rightarrow \frac{du}{dx} = 2x - 7$$

$$y = (x^2 - 7x + 1)^5 \Rightarrow \frac{dy}{dx} = 5(x^2 - 7x + 1)^4(2x - 7)$$

$$\Rightarrow \frac{dy}{dx} = (10x - 35)(x^2 - 7x + 1)^4$$

$$\Rightarrow \left(\frac{dy}{dx}\right)_{x=0} = (10(0) - 35)((0)^2 - 7(0) + 1)^4 = (-35)(1)^4 = -35$$

7 (c) (i)

Draw up a *s*, *v*, *a* table as shown on the right.
The first firework exploded after 5 seconds. You are asked to calculate the height *h* after a time *t* = 5 s.

$$v = \frac{ds}{dt}$$

$$a = \frac{dv}{dt}$$

$$h = 80t - 5t^2$$
$$v = \frac{dh}{dt} = 80 - 10t$$
$$a = \frac{dv}{dt} = -10$$

$$h = 80t - 5t^2 \Rightarrow h = 80(5) - 5(5)^2$$
$$= 400 - 5 \times 25 = 400 - 125$$
$$\therefore h = 275 \text{ m}$$

7 (c) (ii)

You are asked to calculate the speed *v* of the first firework after a time *t* = 5 s.

$$v = 80 - 10t \Rightarrow v = 80 - 10(5) = 80 - 50$$
$$\therefore v = 30 \text{ m/s}$$

7 (c) (iii)

The second firework reaches its maximum height when its velocity is zero. You need to find out the time *t* it takes for its velocity *v* to be zero.

$$v = 80 - 10t \Rightarrow 0 = 80 - 10t$$
$$\Rightarrow 10t = 80$$
$$\therefore t = 8 \text{ s}$$

8 (a) Let $g(x) = x^4 - 32x$ for $x \in \mathbf{R}$.

(i) Write down $g'(x)$, the derivative of $g(x)$.

(ii) For what value of x is $g'(x) = 0$?

(b) Differentiate $3x^2 - x$ from first principles with respect to x.

(c) Let $f(x) = \dfrac{1}{x+1}$ for $x \in \mathbf{R}$ and $x > -1$.

(i) Find $f'(x)$.

(ii) Find the co-ordinates of the point on the curve of $f(x)$ at which the tangent has slope of $-\frac{1}{4}$.

(iii) Find the equation of the tangent to the curve which has slope of $-\frac{1}{4}$.

SOLUTION

8 (a) (i)

$$\boxed{y = x^n \Rightarrow \frac{dy}{dx} = nx^{n-1}}$$

CONSTANT RULE: If y = Constant $\Rightarrow \dfrac{dy}{dx} = 0$

MULTIPLY BY A CONSTANT RULE: If $y = cu$, where c is a constant and u is a function of x, $\dfrac{dy}{dx} = c \times \dfrac{du}{dx}$.

$g(x) = x^4 - 32x$

$\Rightarrow g'(x) = 4x^3 - 32$

8 (a) (ii)

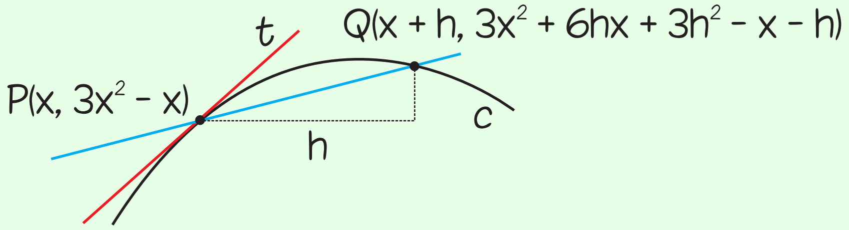
$g'(x) = 0 \Rightarrow 4x^3 - 32 = 0$

$\Rightarrow 4x^3 = 32$

$\Rightarrow x^3 = 8$

$\therefore x = 2$

8 (b)



$$y = 3(x)^2 - (x)$$
$$x = x : y = 3x^2 - x$$

$$y = 3(x)^2 - (x)$$
$$\begin{aligned} x = x+h : y &= 3(x+h)^2 - (x+h) \\ &= 3(x+h)(x+h) - (x+h) \\ &= 3(x^2+hx+hx+h^2) - (x+h) \\ &= 3(x^2+2hx+h^2) - (x+h) \\ &= 3x^2+6hx+3h^2-x-h \end{aligned}$$

$$\begin{array}{ll} Q(\overset{x_2}{x+h}, & \overset{y_2}{3x^2+6hx+3h^2-x-h}) \\ \quad\downarrow & \quad\downarrow \\ P(\underset{}{\overset{x_1}{x}}, & \overset{y_1}{3x^2-x}) \end{array}$$

$$\boxed{m = \frac{y_2 - y_1}{x_2 - x_1}}$$

$$\begin{aligned} \text{Slope of } PQ &= \frac{(3x^2+6hx+3h^2-x-h)-(3x^2-x)}{x+h-x} \\ &= \frac{3x^2+6hx+3h^2-x-h-3x^2+x}{x+h-x} \\ &= \frac{6hx+3h^2-h}{h} \\ &= \frac{h(6x+3h-1)}{h} = 6x+3h-1 \end{aligned}$$

$$\frac{dy}{dx} = \lim_{h \to 0}(6x+3h-1) = 6x-1$$

8 (c) (i)

$$f(x) = \frac{1}{(x+1)} = (x+1)^{-1}$$

$$\Rightarrow f'(x) = -1(x+1)^{-2}(1) = -\frac{1}{(x+1)^2}$$

8 (c) (ii)

Finding the coordinates of the point of contact of the tangent given the slope:

Steps: Finding points of contact given the slope, m, of the tangent

1. Find $\dfrac{dy}{dx}$.

2. Put $\dfrac{dy}{dx} = m$ and solve for x.

3. Find the corresponding y values.

1. $y = f(x) = \dfrac{1}{(x+1)}$

$$\Rightarrow \frac{dy}{dx} = f'(x) = -\frac{1}{(x+1)^2}$$

2. $\dfrac{dy}{dx} = -\dfrac{1}{4} \Rightarrow -\dfrac{1}{(x+1)^2} = -\dfrac{1}{4}$

$\Rightarrow (x+1)^2 = 4$

$\Rightarrow (x+1) = \pm 2$

$\therefore x = 1, -3$ [As $x > -1$ ignore one of the solutions.]

$\therefore x = 1$

3. $x = 1$: $y = f(1) = \dfrac{1}{(1+1)} = \frac{1}{2} \Rightarrow (1, \frac{1}{2})$ is the point of contact.

8 (c) (iii)

Point $(1, \frac{1}{2})$, $m = -\frac{1}{4}$ Equation of a line: $\boxed{y - y_1 = m(x - x_1)}$

$y - \frac{1}{2} = -\frac{1}{4}(x-1)$

$\Rightarrow 4(y - \frac{1}{2}) = -1(x-1)$

$\Rightarrow 4y - 2 = -1x + 1$

$\Rightarrow x + 4y - 3 = 0$