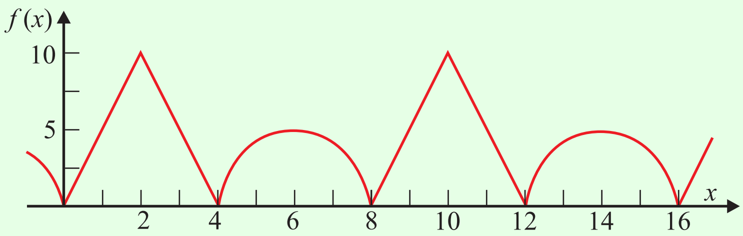
DIFFERENTIATION & FUNCTIONS (Q 6, 7 & 8, PAPER 1)

2000

6 (a) Differentiate $7x + 3$ from first principles with respect to x.

(b)



The graph shows portion of a periodic function $f : x \to f(x)$ which is defined for $x \in \mathbf{R}$.

(i) Write down the period and the range of $f(x)$.

(ii) Complete the following table:

x	2	8	14	20	26
$f(x)$					

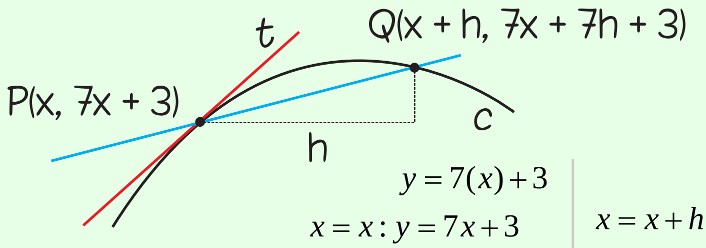
(c) Let $g(x) = (2x+3)(x^2 - 1)$ for $x \in \mathbf{R}$.

(i) For what two values of x is the slope of the tangent to the curve of $g(x)$ equal to 10?

(ii) Find the equations of the two tangents to the curve of $g(x)$ which have slope 10.

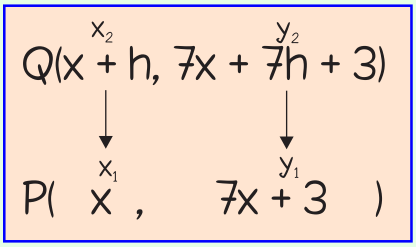
SOLUTION

6 (a)



$$y = 7(x) + 3$$
$$x = x : y = 7x + 3$$

$$y = 7(x) + 3$$
$$x = x + h : y = 7(x+h) + 3$$
$$= 7x + 7h + 3$$



$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$\text{Slope of } PQ = \frac{(7x+7h+3)-(7x+3)}{x+h-x}$$
$$= \frac{7x+7h+3-7x-3}{x+h-x}$$
$$= \frac{7h}{h} = 7$$

$$\frac{dy}{dx} = \lim_{h \to 0}(7) = 7$$

6 (b) (i)
Period = 8
Range = [0, 10]

Period: Length of the wave along the x-axis before it repeats itself.

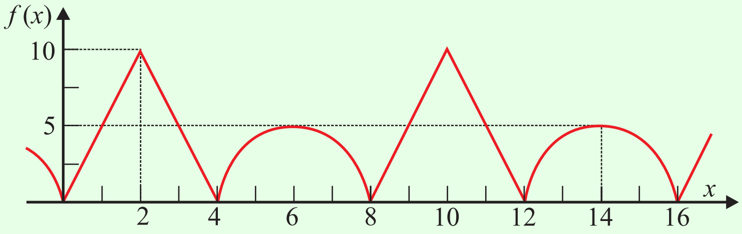
Range: The interval between the lowest y value and the highest y value.

6 (b) (ii)
The first 3 values can be worked out from the graph as shown.

$f(2)=10$

$f(8)=0$

$f(14)=5$



The value of the function at any value of x can be worked out from the first wave by dividing the value of x by the period and finding the remainder.

$$f(x) = f(\text{Remainder})$$

The last 2 values are worked out using the information as explained in the box above. Divide the value of the function by the period and take the remainder.

$f(20)=f(4)=0$

$f(26)=f(2)=10$

x	2	8	14	20	26
$f(x)$	10	0	5	0	10

6 (c) (i)

You need to differentiate the function $g(x)$. You can multiply it out and differentiate term by term or you can use the product rule. Here, we multiply it out.

$$g(x)=(2x+3)(x^2-1)=2x^3-2x+3x^2-3$$

$$\Rightarrow g(x)=2x^3+3x^2-2x-3$$

$$\therefore g'(x)=2\times 3x^2+3\times 2x-2-0$$

$$\Rightarrow g'(x)=6x^2+6x-2$$

$$g(x)=(2x+3)(x^2-1)$$

$$g'(x)=6x^2+6x-2$$

$$g'(x)=10 \Rightarrow 6x^2+6x-2=10$$

$$\Rightarrow 6x^2+6x-12=0$$

$$\Rightarrow x^2+x-2=0$$

$$\Rightarrow (x+2)(x-1)=0$$

$\therefore x=-2, 1$ [Only the x values are required for part **(i)**. However, the y values are required for part **(ii)**.]

$x=-2: y=g(-2)=(2(-2)+3)((-2)^2-1)=(-1)(3)=-3 \Rightarrow (-2, -3)$ is a point of contact.

$x=1: y=g(1)=(2(1)+3)((1)^2-1)=(5)(0)=0 \Rightarrow (1, 0)$ is a point of contact.

6 (c) (ii)

Equation of a line: $\boxed{y - y_1 = m(x - x_1)}$

TANGENT 1: Point $(-2, -3)$; $m = 10$

$\therefore (y - (-3)) = 10(x - (-2))$

$\Rightarrow y + 3 = 10(x + 2)$

$\Rightarrow y + 3 = 10x + 20$

$\Rightarrow 10x - y + 17 = 0$

TANGENT 2: Point (1, 0); m = 10

$\therefore (y - 0) = 10(x - 1)$

$\Rightarrow y = 10x - 10$

$\Rightarrow 10x - y - 10 = 0$

7 (a) Differentiate with respect to x

(i) $4x^2+5$

(ii) $9x-x^3$.

(b) (i) Find $\frac{dy}{dx}$ when $y=\frac{2x-7}{x-1},\ x\neq 1$.

(ii) Find $\frac{dy}{dx}$ when $y=(x^2+5x-1)^3$.

(c) A car, starting at $t = 0$ seconds, travels a distance of s metres in t seconds where

$$s=30t-\tfrac{9}{4}t^2.$$

(i) Find the speed of the car after 2 seconds.

(ii) After how many seconds is the speed of the car equal to zero?

(iii) Find the distance travelled by the car up to the time its speed is zero.

SOLUTION

7 (a) (i)

$$\boxed{y=x^n \Rightarrow \frac{dy}{dx}=nx^{n-1}}$$

CONSTANT RULE: If y = Constant $\Rightarrow \frac{dy}{dx}=0$

MULTIPLY BY A CONSTANT RULE: If $y = cu$, where c is a constant and u is a function of x, $\frac{dy}{dx}=c\times\frac{du}{dx}$.

$$y=4x^2+5$$

$$\Rightarrow \frac{dy}{dx}=4\times 2x+0=8x$$

7 (a) (ii)

$$y=9x-x^3$$

$$\Rightarrow \frac{dy}{dx}=9-3x^2$$

7 (b) (i)

$$\boxed{\begin{aligned} u=(2x-7)&\Rightarrow \frac{du}{dx}=2 \\ v=(x-1)&\Rightarrow \frac{dv}{dx}=1 \end{aligned}} \qquad \boxed{\frac{dy}{dx}=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}}$$

$$y=\frac{2x-7}{x-1}$$

$$\Rightarrow \frac{dy}{dx}=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}=\frac{(x-1)2-(2x-7)1}{(x-1)^2}$$

$$\Rightarrow \frac{dy}{dx}=\frac{2x-2-2x+7}{(x-1)^2}=\frac{5}{(x-1)^2}$$

7 (b) (ii)

Move the power down in front of the bracket.
Take one away from the power.
Multiply by the differentiation of the inside of the bracket.

$$u=(x^2+5x-1)\Rightarrow\frac{du}{dx}=2x+5$$

$$y=(x^2+5x-1)^3$$

$$\Rightarrow\frac{dy}{dx}=3(x^2+5x-1)^2(2x+5)$$

$$\Rightarrow\frac{dy}{dx}=(6x+15)(x^2+5x-1)^2$$

7 (c) (i)

Draw up a *s*, *v*, *a* table as shown on the right.
you are asked to find the speed *v* after a time
t = 2 seconds.

$$v=30-\tfrac{9}{2}t\Rightarrow v=30-\tfrac{9}{2}(2)=30-9$$

$$\therefore v=21\text{ m/s}$$

$$v=\frac{ds}{dt}$$

$$a=\frac{dv}{dt}$$

$$s=30t-\tfrac{9}{4}t^2$$

$$v=\frac{ds}{dt}=30-\tfrac{9}{4}\times 2t=30-\tfrac{9}{2}t$$

$$a=\frac{dv}{dt}=-\tfrac{9}{2}$$

7 (c) (ii)

You are asked to find the time *t* when the speed *v* is zero.

$$v=30-\tfrac{9}{2}t\Rightarrow 0=30-\tfrac{9}{2}t$$

$$\Rightarrow 30=\tfrac{9}{2}t\Rightarrow 60=9t$$

$$\therefore t=\tfrac{60}{9}=\tfrac{20}{3}\text{ s}$$

7 (c) (iii)

You are asked to find the distance *s* travelled after a time $t=\tfrac{20}{3}$ s.

$$s=30t-\tfrac{9}{4}t^2\Rightarrow s=30(\tfrac{20}{3})-\tfrac{9}{4}(\tfrac{20}{3})^2$$

$$\Rightarrow s=10(20)-\tfrac{9}{4}(\tfrac{400}{9})$$

$$\Rightarrow s=200-100$$

$$\therefore s=100\text{ m}$$

8 (a) Let $p(x) = 3x - 12$.
For what values of x is $p(x) < 0$ where x is a positive whole number?

(b) (i) Draw the graph of

$$g(x) = \tfrac{1}{x} \text{ for } -3 \le x \le 3,\ x \in \mathbf{R} \text{ and } x \ne 0.$$

(ii) Using the same axes and the same scales, draw the graph of

$$h(x) = x + 1 \text{ for } -3 \le x \le 3,\ x \in \mathbf{R}.$$

(iii) Use your graphs to estimate the values of x for which

$$\tfrac{1}{x} = x + 1.$$

(c) Let $f(x) = x^3 - 3x^2 + ax + 1$ for all $x \in \mathbf{R}$ and for $a \in \mathbf{R}$.
$f(x)$ has a turning point (a local maximum or a local minimum) at $x = -1$.
(i) Find the value of a.
(ii) Is this turning point a local maximum or a local minimum?
Give a reason for your answer.
(iii) Find the co-ordinates of the other turning point of $f(x)$.

Solution

8 (a)

$p(x) = 3x - 12$

$\therefore 3x - 12 < 0$

$\Rightarrow 3x < 12$

$\Rightarrow x < 4$ [The whole positive numbers less than 4 are 1, 2 and 3.]

$\therefore x = \{1, 2, 3\}$

8 (b) (i)

$x = -3: f(-3) = \dfrac{1}{-3} = -\dfrac{1}{3} = -0.33 \Rightarrow (-3, -0.33)$

$x = -2: f(-2) = \dfrac{1}{-2} = -\dfrac{1}{2} = -0.5 \Rightarrow (-2, -0.5)$

$x = -1: f(-1) = \dfrac{1}{-1} = -1 \Rightarrow (-1, -1)$

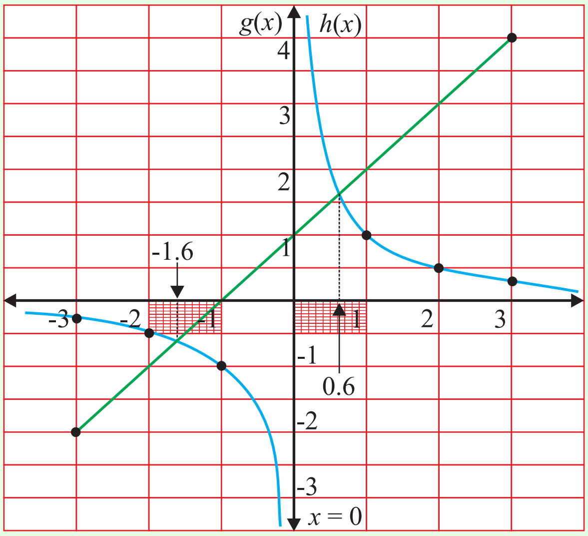
$x = 0:$ This is the gap where the function is not defined.

$x = 1: f(1) = \dfrac{1}{1} = 1 \Rightarrow (1, 1)$

$x = 2: f(2) = \dfrac{1}{2} = 0.5 \Rightarrow (2, 0.5)$

$x = 3: f(3) = \dfrac{1}{3} = 0.33 \Rightarrow (3, 0.33)$

x	$g(x)$
–3	–0.33
–2	0.5
–1	–1
0	Error
1	1
2	0.5
3	0.33



8 (b) (ii)

As these graphs are straight lines, you only need to plot the first and last points in the domain.

x	$h(x)$
-3	-2
3	4

$h(x)=x+1$

$\Rightarrow h(-3)=-3+1=-2 \Rightarrow (-3,-2)$ is a point on the straight line.

$\Rightarrow h(3)=3+1=4 \Rightarrow (3,\ 4)$ is a point on the straight line.

8 (b) (iii)

Read off the x values of where the graphs intersect.

$\therefore x=-1.6,\ 0.6$

8 (c) (i)

$$\boxed{\text{Turning Point: } \frac{dy}{dx}=0}$$

You are told there is a turning point at $x=-1$. Therefore, you need to find $\frac{dy}{dx}$ at $x=-1$ and set the answer equal to zero as it is a turning point.

$y=f(x)=x^3-3x^2+ax+1$

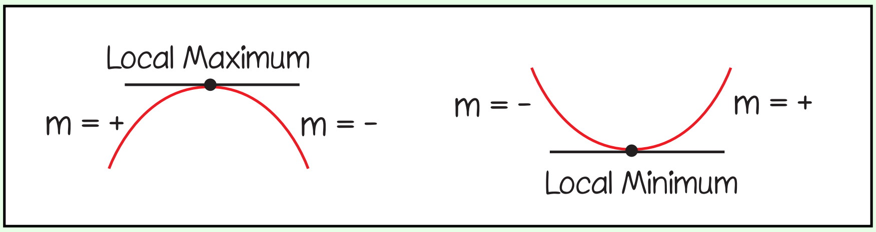
$\therefore \frac{dy}{dx}=3x^2-6x+a$

$\therefore \left(\frac{dy}{dx}\right)_{x=-1}=3(-1)^2-6(-1)+a=0$

$\Rightarrow 3+6+a=0$

$\therefore a=-9$

8 (c) (ii)



The turning point occurs at x = –1. Find the slope just before this (*x* = –2) and just after (*x* = 0)

$$y = f(x) = x^3 - 3x^2 - 9x + 1$$

$$\Rightarrow \frac{dy}{dx} = 3x^2 - 6x - 9$$

$$\left(\frac{dy}{dx}\right)_{x=-2} = 3(-2)^2 - 6(-2) - 9 = 12 + 12 - 9 = 15$$

$$\left(\frac{dy}{dx}\right)_{x=0} = 3(0)^2 - 6(0) - 9 = 0 + 0 - 9 = -9$$

As the slope changes from positive to negative, the turning point is a local maximum.

8 (c) (iii)

$$y = f(x) = x^3 - 3x^2 - 9x + 1$$

$$\Rightarrow \frac{dy}{dx} = 3x^2 - 6x - 9$$

$$\frac{dy}{dx} = 0 \Rightarrow 3x^2 - 6x - 9 = 0$$

$$\boxed{\text{Turning Point: } \frac{dy}{dx} = 0}$$

$$\Rightarrow x^2 - 2x - 3 = 0$$

$$\Rightarrow (x-3)(x+1) = 0$$

$$\therefore x = -1,\ 3$$

Anyway, it is the turning point at $x = 3$ which is of interest.

$$\therefore y = f(3) = (3)^3 - 3(3)^2 - 9(3) + 1 = 27 - 27 - 27 + 1 = -26$$

Therefore, $(3, -26)$ is the other turning point.