DIFFERENTIATION & FUNCTIONS (Q 6, 7 & 8, PAPER 1)

1999

6 (a) Let $f(x) = 2(3x-1),\ x \in \mathbf{R}$.
Find the value of x for which $f(x) = 0$.

(b) Differentiate from first principles

$$x^2 + 5x$$

with respect to x.

(c) Let $f(x) = x^3 - 6x^2 + 12$ for $x \in \mathbf{R}$.
Find the derivative of $f(x)$.
At the two points (x_1, y_1) and (x_2, y_2), the tangents to the curve $y = f(x)$ are parallel to the x axis, where $x_2 > x_1$.
Show that

(i) $x_2 - x_1 = 4$

(ii) $y_2 = y_1 - 32$.

SOLUTION

6 (a)

$f(x) = 2(3x-1)$

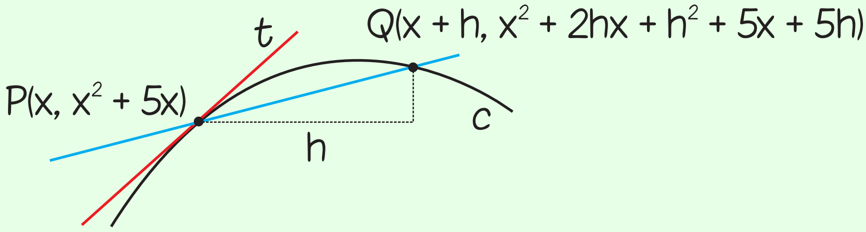
$f(x) = 0 \Rightarrow 2(3x-1) = 0$

$\Rightarrow 3x - 1 = 0$

$\Rightarrow 3x = 1$

$\therefore x = \frac{1}{3}$

6 (b)



$y = (x)^2 + 5(x)$

$x = x : y = x^2 + 5x$

$y = (x)^2 + 5(x)$

$x = x + h : y = (x+h)^2 + 5(x+h)$

$y = (x+h)(x+h) + 5(x+h)$

$y = (x^2 + hx + hx + h^2) + 5(x+h)$

$y = x^2 + 2hx + h^2 + 5x + 5h$

$$Q(\overset{x_2}{x+h},\ \overset{y_2}{x^2+2hx+h^2+5x+5h})$$

$$P(\underset{x_1}{x},\ \underset{y_1}{x^2+5x})$$

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Slope of PQ $= \dfrac{(x^2+2hx+h^2+5x+5h)-(x^2+5x)}{x+h-x}$

$$= \frac{x^2+2hx+h^2+5x+5h-x^2-5x}{x+h-x}$$

$$= \frac{2hx+h^2+5h}{h}$$

$$= \frac{h(2x+h+5)}{h}$$

$$= 2x+h+5$$

$$\frac{dy}{dx} = \lim_{h\to 0}(2x+h+5) = 2x+5$$

6 (c)

$$f(x) = x^3-6x^2+12$$

$$\Rightarrow f'(x) = 3x^2-12x$$

$$\boxed{y = x^n \Rightarrow \frac{dy}{dx} = nx^{n-1}}$$

As the tangents are parallel to the x-axis, their slopes are zero.

$$y = f(x) = x^3-6x^2+12$$

$$\Rightarrow \frac{dy}{dx} = f'(x) = 3x^2-12x$$

$$\frac{dy}{dx} = 0 \Rightarrow 3x^2-12x = 0$$

$$\Rightarrow 3x(x-4) = 0$$

$$\therefore x = 0,\ 4$$

$x=0: y = f(0) = (0)^3-6(0)^2+12 = 12 \Rightarrow (0,\ 12)$ is a point of contact.

$x=4: y = f(4) = (4)^3-6(4)^2+12 = 64-96+12 \Rightarrow (4,\ -20)$ is a point of contact.

First point: $(x_1,\ y_1) = (0,\ 12)$

Second point: $(x_2,\ y_2) = (4,\ -20)$

6 (c) (i)

$x_2 - x_1 = 4-0 = 4$ [This is true.]

6 (c) (ii)

$$y_2 = y_1 - 32$$

$\Rightarrow -20 = 12-32$ [This is true.]

7 (a) Differentiate

$$2x^3 - 7$$

with respect to x.

(b) (i) Find $\frac{dy}{dx}$ when $y = (3-7x)^5$.

(ii) Find $\frac{dy}{dx}$ when $y = \frac{x^2}{1-x}$, $x \neq 1$. Show that $\frac{dy}{dx} = 0$ at $x = 0$.

(c) The speed, v, in metres per second, of a body after t seconds is given by

$$v = 3t(4-t).$$

(i) Find the acceleration at each of the two instants when the speed is 9 metres per second.

(ii) Find the speed at the instant when the acceleration is zero.

SOLUTION

7 (a)

$$\boxed{y = x^n \Rightarrow \frac{dy}{dx} = nx^{n-1}}$$

CONSTANT RULE: If y = Constant $\Rightarrow \frac{dy}{dx} = 0$

MULTIPLY BY A CONSTANT RULE: If $y = cu$, where c is a constant and u is a function of x, $\frac{dy}{dx} = c \times \frac{du}{dx}$.

$$y = 2x^3 - 7$$

$$\Rightarrow \frac{dy}{dx} = 2 \times 3x^2 - 0 = 6x^2$$

7 (b) (i)

Move the power down in front of the bracket.
Take one away from the power.
Multiply by the differentiation of the inside of the bracket.

$$\boxed{u = 3 - 7x \Rightarrow \frac{du}{dx} = -7}$$

$$y = (3-7x)^5 \Rightarrow \frac{dy}{dx} = 5(3-7x)^4(-7) = -35(3-7x)^4$$

7 (b) (ii)

$$u = x^2 \Rightarrow \frac{du}{dx} = 2x$$
$$v = (1-x) \Rightarrow \frac{dv}{dx} = -1$$

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$y = \frac{x^2}{1-x}$$
$$\Rightarrow \frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2} = \frac{(1-x)(2x) - x^2(-1)}{(1-x)^2}$$
$$\Rightarrow \frac{dy}{dx} = \frac{2x - 2x^2 + x^2}{(1-x)^2} = \frac{2x - x^2}{(1-x)^2}$$
$$\therefore \left(\frac{dy}{dx}\right)_{x=0} = \frac{2(0)-(0)^2}{(1-0)^2} = \frac{0}{1} = 0 \text{ [This is true.]}$$

7 (c) (i)

Draw up a v, a table as shown on the right.
Firstly, find the times t when the speed v = 9 m/s.
Then, find the accelerations a at these times.

$$v = \frac{ds}{dt}$$
$$a = \frac{dv}{dt}$$

$$v = 3t(4-t) = 12t - 3t^2$$
$$a = \frac{dv}{dt} = 12 - 6t$$

$$v = 12t - 3t^2 \Rightarrow 9 = 12t - 3t^2$$
$$\Rightarrow 3t^2 - 12t + 9 = 0$$
$$\Rightarrow t^2 - 4t + 3 = 0$$
$$\Rightarrow (t-1)(t-3) = 0$$
$$\therefore t = 1 \text{ s, } 3 \text{ s}$$
$$t = 1 : a = 12 - 6(1) = 12 - 6 = 6 \text{ m/s}^2$$
$$t = 3 : a = 12 - 6(3) = 12 - 18 = -6 \text{ m/s}^2$$

7 (c) (ii)

Firstly, find the time t at which the acceleration is zero. Then, find the speed v at this time.

$$a = 12 - 6t \Rightarrow 0 = 12 - 6t$$
$$\Rightarrow 6t = 12$$
$$\therefore t = 2 \text{ s}$$
$$t = 2 : v = 3t(4-t) \Rightarrow v = 3(2)(4-2)$$
$$\therefore v = 6(2) = 12 \text{ m/s}$$

8 Let $f(x)=2x^3-5x^2-4x+3$ for $x\in\mathbf{R}$.

(i) Complete the table

x	-1.5	-1	0	1	2	3	3.5
$f(x)$	-9						13.5

(ii) Find the derivative of $f(x)$.
Calculate the co-ordinates of the local minimum and show that the co-ordinates of the local maximum are $(-\frac{1}{3},\ \frac{100}{27})$.

(iii) Draw the graph of

$$f(x)=2x^3-5x^2-4x+3$$

for $-1.5\le x\le 3.5$.

(iv) Write the equation $2x^3-5x^2-6x+6=0$ in the form

$$2x^3-5x^2-4x+3=ax+b,\quad a,\ b\in\mathbf{Z}.$$

Hence, use your graph to estimate the solutions of the equation

$$2x^3-5x^2-6x+6=0.$$

SOLUTION

8 (i)

$f(x)=2x^3-5x^2-4x+3$

$\Rightarrow f(-1.5)=2(-1.5)^3-5(-1.5)^2-4(-1.5)+3=-9$ [Use your calculator.]

Find the other values of x in the same way.

x	-1.5	-1	0	1	2	3	3.5
$f(x)$	-9	0	3	-4	-9	0	13.5

8 (ii)

$y=f(x)=2x^3-5x^2-4x+3$

$\frac{dy}{dx}=f'(x)=6x^2-10x-4$

$\frac{dy}{dx}=0\Rightarrow 6x^2-10x-4=0$

$\Rightarrow 3x^2-5x-2=0$

$\Rightarrow (3x+1)(x-2)=0$

$\therefore x=-\frac{1}{3},\ 2$

Turning Point: $\frac{dy}{dx}=0$

$x=-\frac{1}{3}: y=2(-\frac{1}{3})^3-5(-\frac{1}{3})^2-4(-\frac{1}{3})+3=\frac{100}{27}\Rightarrow(-\frac{1}{3},\ \frac{100}{27})$ is a local maximum.

$x=2: y=2(2)^3-5(2)^2-4(2)+3=-9\Rightarrow(2,\ -9)$ is a local minimum.

[$(-\frac{1}{3},\ \frac{100}{27})$ is the local maximum as its y coordinate is greater than the y coordinate of the other point.]

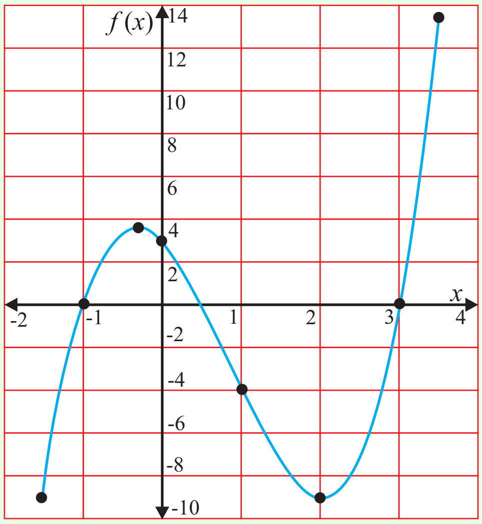
8 (iii)

Draw the cubic graph using the information from the previous parts.

Points: $(-1.5, -9), (-1, 0), (0, 3), (1, -4)$
$(2, -9), (3, 0), (3.5, 13.5)$

Local maximum: $(-\frac{1}{3}, \frac{100}{27}) = (-0.33, 3.7)$

Local minimum: $(2, -9)$



8 (iv)

$2x^3 - 5x^2 - 6x + 6 = 0$

$\Rightarrow 2x^3 - 5x^2 - 4x + 3 = 2x - 3$

Let $h(x) = 2x - 3$

$\therefore f(x) = h(x)$.

$h(x)$ is a straight line. You want to find where the straight line and the cubic graph intersect. Graph $h(x)$ by using the first and last points of the domain.

$h(-1.5) = 2(-1.5) - 3 = -3 - 3 = -6$

$\Rightarrow (-1.5, -6)$ is a point on the graph.

$h(3.5) = 2(3.5) - 3 = 7 - 3 = 4$

$\Rightarrow (3.5, 4)$ is a point on the graph.

You can see the x values of the places where the graphs intersect:

$\therefore x = -1.4, 0.7, 3.2$