Differentiation & Functions (Q 6, 7 & 8, Paper 1)

1998

6 (a) If $f(x) = 5x - 8$ and $g(x) = 13 - 2x$, find the value of x for which

$$f(x) = g(x).$$

(b) The speed, v, in metres per second of an engine moving along a track is related to time, t, in seconds by

$$v = \tfrac{1}{3}(2t + 5).$$

(i) Draw the straight line graph of this relation, putting t on the horizontal axis, for $0 \le t \le 8$.

(ii) Use your graph to estimate the speed when $t = 2.5$ seconds.

(iii) Use your graph to estimate the time at which the speed reaches 6 metres per second.

(c) $f(x) = (x + k)(x - 2)^2$, where k is a real number.

(i) If $f(3) = 7$, find the value of k.

(ii) Using this value for k, find the coordinates of the local maximum and of the local minimum of $f(x)$.

Solution

6 (a)

$f(x) = g(x)$

$\Rightarrow 5x - 8 = 13 - 2x$

$\Rightarrow 5x + 2x = 13 + 8$

$\Rightarrow 7x = 21$

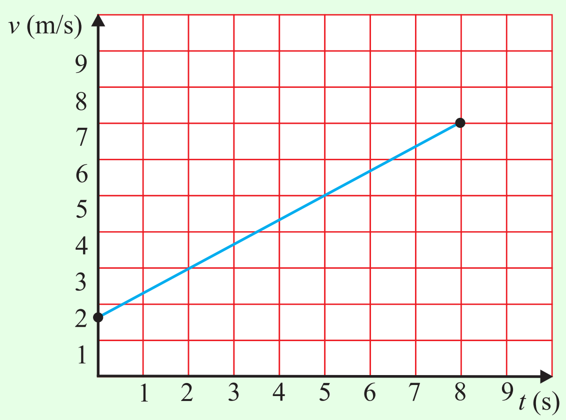
$\therefore x = 3$

6 (b) (i)

As these graphs are straight lines, you only need to plot the first and last points in the domain.

$t = 0 : v = \tfrac{1}{3}(2t + 5) = \tfrac{1}{3}(2(0) + 5) = \tfrac{1}{3}(5) = \tfrac{5}{3} \Rightarrow (0, \tfrac{5}{3})$ is a point on the line.

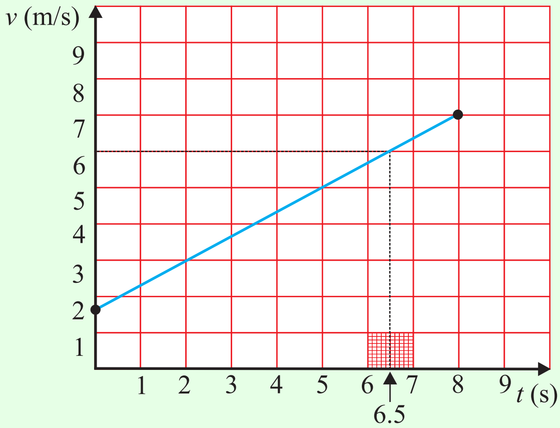
$t = 8 : v = \tfrac{1}{3}(2t + 5) = \tfrac{1}{3}(2(8) + 5) = \tfrac{1}{3}(21) = 7 \Rightarrow (8, 7)$ is a point on the line.



6 (b) (ii)

Start at 2.5 s along the horizontal axis and go straight up till you meet the graph. Now go straight across to the vertical axis and read off the speed *v*.

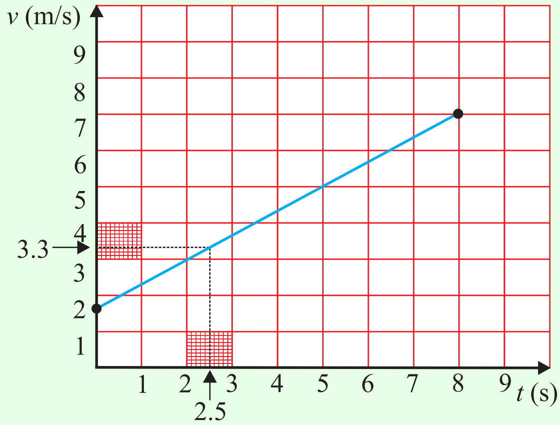
$\therefore v = 3.3$ m/s



6 (b) (iii)

Start at 6 m/s along the vertical axis and go straight across till you meet the graph. Now go straight down to the horizontal axis and read off the time *t*.

$\therefore t = 6.5$ s



6 (c) (i)

$f(x) = (x+k)(x-2)^2$

$f(3) = 7 \Rightarrow (3+k)(3-2)^2 = 7$

$\Rightarrow (3+k)(1)^2 = 7$

$\Rightarrow 3+k = 7$

$\therefore k = 4$

6 (c) (ii)

$f(x) = (x+4)(x-2)^2$ [Multiply this out and tidy up.]

$\Rightarrow f(x) = (x+4)(x^2-4x+4)$

$\Rightarrow f(x) = x^3 - 4x^2 + 4x + 4x^2 - 16x + 16$

$\therefore f(x) = x^3 - 12x + 16$

$y = f(x) = x^3 - 12x + 16$

$\frac{dy}{dx} = f'(x) = 3x^2 - 12$

$\boxed{\text{Turning Point: } \frac{dy}{dx} = 0}$

$\frac{dy}{dx} = 0 \Rightarrow 3x^2 - 12 = 0$

$\Rightarrow 3(x^2 - 4) = 0$ $\boxed{a^2 - b^2 = (a+b)(a-b)}$

$\Rightarrow 3(x+2)(x-2) = 0$

$\therefore x = -2, 2$

$x = -2: y = f(-2) = (-2)^3 - 12(-2) + 16 = -8 + 24 + 16 = 32 \Rightarrow (-2, 32)$ is a local maximum.

$x = 2: y = f(2) = (2)^3 - 12(2) + 16 = 8 - 24 + 16 = 0 \Rightarrow (2, 0)$ is a local minimum.

[(–2, 32) is the local maximum as its *y* coordinate is greater than the *y* coordinate of the other point.]

7 (a) Differentiate with respect to x

(i) $x^2 - 3x$

(ii) $\frac{1}{x^2}$.

(b) (i) Find $\frac{dy}{dx}$ when $y = \frac{2x}{x^2+1}$.

(ii) Find $\frac{dy}{dx}$ when $y = (4-3x^2)^7$ and write down the range of values of x for which $\frac{dy}{dx} > 0$.

(c) The volume of water, V, in cm^3, that remains in a leaking tank after t seconds is given by

$$V = 45000 - 300t + 0.5t^2.$$

(i) After how many seconds will the tank be empty?

(ii) Find the rate of change of the volume with respect to t when t = 50 seconds.

SOLUTION

7 (a) (i)

$$\boxed{y = x^n \Rightarrow \frac{dy}{dx} = nx^{n-1}}$$

CONSTANT RULE: If y = Constant $\Rightarrow \frac{dy}{dx} = 0$

MULTIPLY BY A CONSTANT RULE: If $y = cu$, where c is a constant and u is a function of x, $\frac{dy}{dx} = c \times \frac{du}{dx}$.

$$y = x^2 - 3x$$

$$\Rightarrow \frac{dy}{dx} = 2x - 3$$

7 (a) (ii)

$$y = \frac{1}{x^2} = x^{-2}$$

$$\Rightarrow \frac{dy}{dx} = -2x^{-3} = -\frac{2}{x^3}$$

$\boxed{a^{-n} = \frac{1}{a^n}}$ **...Power Rule No. 4**

7 (b) (i)

$$u = 2x \Rightarrow \frac{du}{dx} = 2$$
$$v = (x^2+1) \Rightarrow \frac{dv}{dx} = 2x$$

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$y = \frac{2x}{x^2+1}$$

$$\Rightarrow \frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2} = \frac{(x^2+1)2 - 2x(2x)}{(x^2+1)^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{2x^2+2-4x^2}{(x^2+1)^2} = \frac{2-2x^2}{(x^2+1)^2}$$

7 (b) (i)

Move the power down in front of the bracket.
Take one away from the power.
Multiply by the differentiation of the inside of the bracket.

$$u = (4-3x^2) \Rightarrow \frac{du}{dx} = -6x$$

$$y = (4-3x^2)^7 \Rightarrow \frac{dy}{dx} = 7(4-3x^2)^6(-6x) = -42x(4-3x^2)^6$$

$$\frac{dy}{dx} > 0 \Rightarrow -42x(4-3x^2)^6 > 0$$

What values of x make this statement true?

$(4-3x^2)^6 > 0$ is true for all values of x. No matter what value is inside the bracket, once you raise it to an even power the value will be positive.

This means that x must be negative so that the $-42x$ is also positive.
Therefore, $x < 0$.

7 (c) (i)

After what time t with the volume $V = 0$?

$$V = 45000 - 300t + 0.5t^2 \Rightarrow 0 = 45000 - 300t + 0.5t^2$$
$$\Rightarrow t^2 - 600t + 90000 = 0$$
$$\Rightarrow (t-300)(t-300) = 0$$
$$\therefore t = 300 \text{ s}$$

7 (c) (ii)

You need to find the rate of change of the volume, $\frac{dV}{dt}$, at a time $t = 50$ s, $\left(\frac{dV}{dt}\right)_{t=50}$.

$$V = 45000 - 300t + 0.5t^2 \Rightarrow \frac{dV}{dt} = -300 + 0.5 \times 2t = -300 + t$$

$$\therefore \left(\frac{dV}{dt}\right)_{t=50} = -300 + 50 = -250 \text{ cm}^3/\text{s}$$

8 Let $f(x)=\dfrac{1}{x-1}$, for $x\in\mathbf{R}$ and $x\neq 1$.

(i) Find the value of $f(-2)$, $f(0)$, $f(\frac{3}{2})$ and $f(5)$.

(ii) Find $f'(x)$, the derivative of $f(x)$.

(iii) Draw the graph of $f(x)$ for $-2\leq x\leq 5$.

(iv) Find the equation of the tangent T to the curve at the point $(0, -1)$.

(v) Find the coordinates of the other point on the graph of $f(x)$ at which the tangent to the curve is parallel to T.

Solution

8 (i)

$$f(x)=\frac{1}{x-1}$$

$$f(-2)=\frac{1}{(-2)-1}=\frac{1}{-3}=-\frac{1}{3}=-0.33$$

$$f(0)=\frac{1}{(0)-1}=\frac{1}{-1}=-1$$

$$f(\tfrac{3}{2})=\frac{1}{(\frac{3}{2})-1}=\frac{1}{\frac{1}{2}}=2$$

$$f(5)=\frac{1}{(5)-1}=\frac{1}{4}=0.25$$

x	$f(x)$
-2	-0.33
0	-1
$\frac{3}{2}$	2
5	0.25

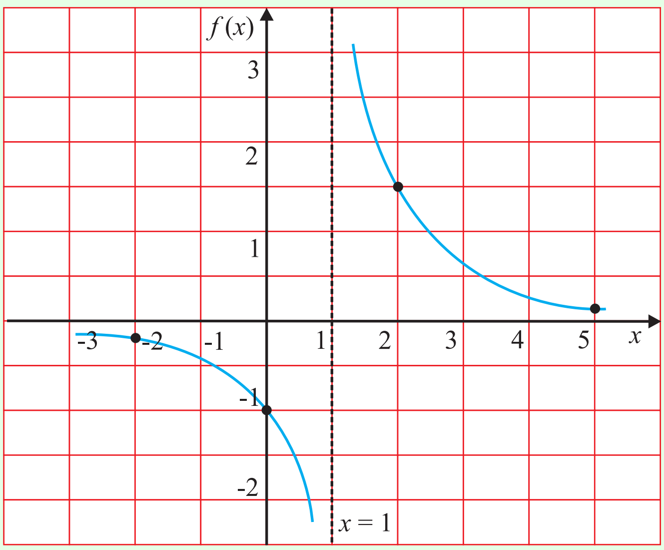
8 (ii)

$$f(x)=\frac{1}{(x-1)}=(x-1)^{-1}$$

$$\Rightarrow f'(x)=-1(x-1)^{-2}=-\frac{1}{(x-1)^2}$$

8 (iii)

Put $(x-1)=0\Rightarrow x=1$ [This is the gap in the graph.]



8 (iv)

$$\frac{dy}{dx}=f'(x)=-\frac{1}{(x-1)^2}$$

$$\left(\frac{dy}{dx}\right)_{x=0}=-\frac{1}{(0-1)^2}=-\frac{1}{(-1)^2}=-1$$

The point of contact is given as $(0,\ -1)$.

Equation of the tangent *T*: Point $(0,\ -1),\ m=-1$

$$T:(y-(-1))=-1(x-0)$$

$$\Rightarrow T: y+1=-x$$

Equation of a line: $\boxed{y-y_1=m(x-x_1)}$

$$\therefore T: x+y+1=0$$

8 (v)

A parallel tangent has the same slope as *T*. $\therefore m=-1$.

$$\frac{dy}{dx}=-\frac{1}{(x-1)^2}$$

$$\frac{dy}{dx}=-1\Rightarrow -\frac{1}{(x-1)^2}=-1$$

$$\Rightarrow 1=(x-1)^2$$

$$\Rightarrow \pm 1=x-1$$

$\therefore x=0,\ 2$ [Ignore the first solution as that was used in part **(iv)**.]

$y=f(2)=\frac{1}{x-1}=\frac{1}{2-1}=\frac{1}{1}=1\Rightarrow(2,\ 1)$ is the other point.