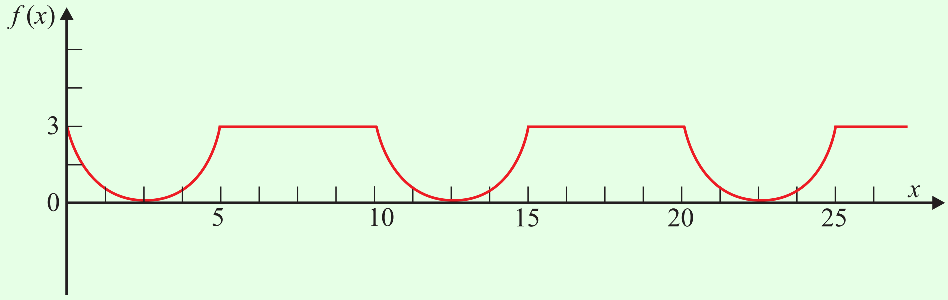
DIFFERENTIATION & FUNCTIONS (Q 6, 7 & 8, PAPER 1)

1997

6 (a)



The graph shows portion of a periodic function $f : x \to f(x)$.
Write down the period and range of the function.
What is the value of $f(77.5)$?

(b) Differentiate from first principles

$$3x^2 - 2$$

with respect to x.

(c) Let $f(x) = ax^3 + bx + c$, for all $x \in \mathbf{R}$ and for $a, b, c \in \mathbf{R}$.
Use the information which follows to find the value of a, of b and of c:
(i) $f(0) = 3$
(ii) the slope of the tangent to the curve of $f(x)$ at $x = 1$ is -18
(iii) the curve of $f(x)$ has a local maximum at $x = 2$.

SOLUTION

6 (a)

PERIOD: Length of the wave along the x-axis before it repeats itself.

RANGE: The interval between the lowest y value and the highest y value.

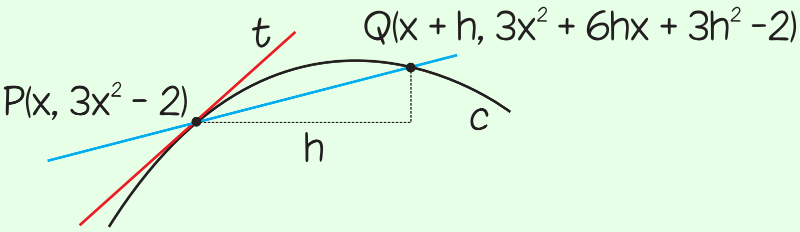
Period = 10
Range = [0, 3]

The value of the function at any value of x can be worked out from the first wave by dividing the value of x by the period and finding the remainder.

$$f(x) = f(\text{Remainder})$$

$f(77.5) = f(7.5) = 3$

6 (b)



$$y = 3(x)^2 - 2$$
$$x = x : y = 3x^2 - 2$$

$$y = 3(x)^2 - 2$$
$$\begin{aligned} x = x + h : y &= 3(x+h)^2 - 2 \\ &= 3(x+h)(x+h) - 2 \\ &= 3(x^2 + hx + hx + h^2) - 2 \\ &= 3(x^2 + 2hx + h^2) - 2 \\ &= 3x^2 + 6hx + 3h^2 - 2 \end{aligned}$$

$$\begin{array}{llll} & \overset{x_2}{} & & \overset{y_2}{} \\ Q(& x + h, & & 3x^2 + 6hx + 3h^2 - 2) \\ & \downarrow & & \downarrow \\ & \overset{x_1}{} & & \overset{y_1}{} \\ P(& x \quad , & & 3x^2 - 2 \qquad) \end{array}$$

$$\boxed{m = \frac{y_2 - y_1}{x_2 - x_1}}$$

$$\begin{aligned} \text{Slope of } PQ &= \frac{(3x^2 + 6hx + 3h^2 - 2) - (3x^2 - 2)}{x + h - x} \\ &= \frac{3x^2 + 6hx + 3h^2 - 2 - 3x^2 + 2}{x + h - x} \\ &= \frac{6hx + 3h^2}{h} = \frac{h(6x + 3h)}{h} \\ &= 6x + 3h \end{aligned}$$

$$\frac{dy}{dx} = \lim_{h \to 0}(6x + 3h) = 6x$$

6 (c) (i)

$$f(x) = ax^3 + bx + c$$
$$f(0) = 3 \Rightarrow a(0)^3 + b(0) + c = 3$$
$$\therefore c = 3$$

6 (c) (ii)

To find the slope of the tangent to the curve at $x = 1$, find $\left(\frac{dy}{dx}\right)_{x=1}$ and put it equal to -18.

$$y = f(x) = ax^3 + bx + 3$$
$$\therefore \frac{dy}{dx} = 3ax^2 + b$$
$$\Rightarrow \left(\frac{dy}{dx}\right)_{x=1} = 3a(1)^2 + b = -18$$
$$\therefore 3a + b = -18 ......\textbf{(1)}$$

6 (c) (iii)

To find the local maximum you put $\frac{dy}{dx}=0$ and solve for x. You are told that you get an answer of x = 2 when you do this.

$\frac{dy}{dx}=0 \Rightarrow 3ax^2+b=0$

$\therefore 3a(2)^2+b=0$

$\Rightarrow 12a+b=0....(\mathbf{2})$

You need to solve equations (**1**) and (**2**) to find a and b.

$$\begin{aligned} 3a+b &= -18..(\mathbf{1})(\times -1) \\ 12a+b &= 0......(\mathbf{2}) \end{aligned} \quad \rightarrow \quad \begin{aligned} -3a-b &= 18 \\ 12a+b &= 0 \\ \hline 9a \quad &= 18 \Rightarrow a=2 \end{aligned}$$

Substitute a = 2 into Equation (**2**).

$\therefore 12(2)+b=0 \Rightarrow 24+b=0$

$\therefore b=-24$

Ans: $a=2,\ b=-24,\ c=3$

7 (a) Differentiate with respect to x

(i) $-x^2$

(ii) $x^4 + x + 1$.

(b) (i) Find $\frac{dy}{dx}$ when $y = (x^2 - 3)(1 - x)$.

(ii) Find the value of $\frac{dy}{dx}$ at $x = -1$ when $y = (3x + 1)^4$.

(c) The distance s metres of an object from a fixed point at t seconds is given by

$$s = \frac{t+1}{t+3}.$$

(i) At what time is the object 0.75 m from a fixed point?

(ii) What is the speed of the object, in terms of t, at t seconds?

(iii) After how many seconds will the speed of the object be less than 0.02 m/s?

SOLUTION

7 (a) (i)

$$\boxed{y = x^n \Rightarrow \frac{dy}{dx} = nx^{n-1}}$$

CONSTANT RULE: If y = Constant $\Rightarrow \frac{dy}{dx} = 0$

MULTIPLY BY A CONSTANT RULE: If $y = cu$, where c is a constant and u is a function of x, $\frac{dy}{dx} = c \times \frac{du}{dx}$.

$$y = -x^2 \Rightarrow \frac{dy}{dx} = -1 \times 2x = -2x$$

7 (a) (ii)

$$y = x^4 + x + 1 \Rightarrow \frac{dy}{dx} = 4x^3 + 1 + 0 = 4x^3 + 1$$

7 (b) (i)

You can multiply out the brackets and differentiate term by term or you can use the product rule.

METHOD 1: Multiply out the brackets.

$$y = (x^2 - 3)(1 - x) = x^2 - x^3 - 3 + 3x$$

$$\therefore y = -x^3 + x^2 + 3x - 3$$

$$\Rightarrow \frac{dy}{dx} = -3x^2 + 2x + 3$$

Method 2: Product rule

$$u=(x^2-3)\Rightarrow\frac{du}{dx}=2x$$
$$v=(1-x)\Rightarrow\frac{dv}{dx}=-1$$

$$\frac{dy}{dx}=u\frac{dv}{dx}+v\frac{du}{dx}$$

$$y=(x^2-3)(1-x)$$
$$\therefore\frac{dy}{dx}=u\frac{dv}{dx}+v\frac{du}{dx}=(x^2-3)(-1)+(1-x)(2x)$$
$$\Rightarrow\frac{dy}{dx}=-x^2+3+2x-2x^2$$
$$\Rightarrow\frac{dy}{dx}=-3x^2+2x+3$$

7 (b) (ii)

Move the power down in front of the bracket.
Take one away from the power.
Multiply by the differentiation of the inside of the bracket.

$$u=3x+1\Rightarrow\frac{du}{dx}=3$$

$$y=(3x+1)^4\Rightarrow\frac{dy}{dx}=4(3x+1)^3(3)=12(3x+1)^3$$
$$\therefore\left(\frac{dy}{dx}\right)_{x=-1}=12(3(-1)+1)^3=12(-3+1)^3$$
$$=12(-2)^3=12(-8)=-96$$

7 (c) (i)

Find the time t at which the distance s = 0.75 m.

$$s=\frac{t+1}{t+3}\Rightarrow 0.75=\frac{t+1}{t+3}$$ [Multiply each side by (t + 3).]
$$\Rightarrow 0.75(t+3)=t+1$$
$$\Rightarrow 0.75t+2.25=t+1$$
$$\Rightarrow 2.25-1=t-0.75t$$
$$\Rightarrow 1.25=0.25t$$
$$\therefore t=\frac{1.25}{0.25}=5\text{ s}$$

7 (c) (ii)

You need to differentiate $s=\dfrac{t+1}{t+3}$ with respect to t to find the speed v. This requires you to use the quotient rule.

$$v=\frac{ds}{dt}$$

$$u=t+1\Rightarrow\frac{du}{dt}=1$$

$$v=t+3\Rightarrow\frac{dv}{dt}=1$$

$$\frac{dy}{dx}=\frac{v\dfrac{du}{dx}-u\dfrac{dv}{dx}}{v^2}$$

$$s=\frac{t+1 \longleftarrow u}{t+3 \longleftarrow v}$$

$$\Rightarrow\frac{ds}{dt}=\frac{v\dfrac{du}{dt}-u\dfrac{dv}{dt}}{v^2}=\frac{(t+3)(1)-(t+1)(1)}{(t+3)^2}$$

$$\Rightarrow\frac{ds}{dt}=\frac{t+3-t-1}{(t+3)^2}=\frac{2}{(t+3)^2}$$

$$\therefore v=\frac{2}{(t+3)^2}$$

7 (c) (iii)

Find out the time t it takes to reach a speed v = 0.02 m/s.

$v=\dfrac{2}{(t+3)^2}\Rightarrow 0.02=\dfrac{2}{(t+3)^2}$ [Multiply across by $(t + 3)^2$.]

$$\Rightarrow 0.02(t+3)^2=2$$

$$\Rightarrow (t+3)^2=\frac{2}{0.02}=100$$

$\Rightarrow (t+3)=\pm 10$ [Take the square root of both sides.]

$\therefore t=7,\ -13$ [Ignore the negative solution as time must by positive.]

Therefore, t = 7 seconds. After 7 seconds, the speed will be less than 0.02 m/s.

8 (a) Let $f(x)=x^2-4x$, for $x\in\mathbf{R}$.

Find $f'(x)$, the derivative of $f(x)$.

For what value of x is $f'(x)=0$?

(b) Find the equation of the tangent to the curve

$$y=x^3-4x+7$$

at the point where $x = 1$.

(c) Draw a graph of

$$g(x)=\frac{1}{x+2}$$

for $0\le x\le 4,\ x\in\mathbf{R}$.

Using the same axes and the same scales draw the graph of

$$h(x)=x-2.$$

Show how your graphs may be used to estimate the value of $\sqrt{5}$.

Solution

8 (a)

$$\boxed{y=x^n\Rightarrow\frac{dy}{dx}=nx^{n-1}}$$

Constant Rule: If y = Constant $\Rightarrow\dfrac{dy}{dx}=0$

Multiply by a constant rule: If $y = cu$, where c is a constant and u is a function of x, $\dfrac{dy}{dx}=c\times\dfrac{du}{dx}$.

$y=f(x)=x^2-4x$

$\Rightarrow\dfrac{dy}{dx}=f'(x)=2x-4$

$f'(x)=0\Rightarrow 2x-4=0$

$\Rightarrow 2x=4$

$\therefore x=2$

8 (b)

$y=f(x)=x^3-4x+7$

$\Rightarrow\dfrac{dy}{dx}=3x^2-4$

$\left(\dfrac{dy}{dx}\right)_{x=1}=3x^2-4=3(1)^2-4=3-4=-1\Rightarrow m=-1$

$x=1: y=f(1)=(1)^3-4(1)+7=1-4+7=4\Rightarrow(x_1,\ y_1)=(1,\ 4)$

$y-y_1=m(x-x_1)$

$$\boxed{y-y_1=m(x-x_1)}$$

$\Rightarrow y-4=-1(x-1)$

$\Rightarrow y-4=-x+1$

$\Rightarrow x+y-5=0$

8 (c)

Put $x+2=0 \Rightarrow x=-2$. This line represents the gap in the graph

$$g(x)=\frac{1}{x+2}$$

$g(0)=\frac{1}{0+2}=\frac{1}{2} \Rightarrow (0, \tfrac{1}{2})$ is a point on the graph.

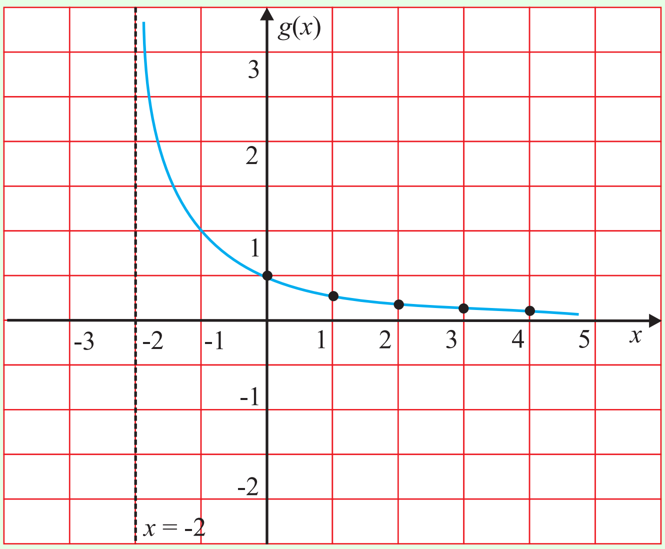
$g(1)=\frac{1}{1+2}=\frac{1}{3} \Rightarrow (0, \tfrac{1}{3})$ is a point on the graph.

$g(2)=\frac{1}{2+2}=\frac{1}{4} \Rightarrow (0, \tfrac{1}{4})$ is a point on the graph.

$g(3)=\frac{1}{3+2}=\frac{1}{5} \Rightarrow (0, \tfrac{1}{5})$ is a point on the graph.

$g(4)=\frac{1}{4+2}=\frac{1}{6} \Rightarrow (0, \tfrac{1}{6})$ is a point on the graph.

x	$f(x)$
0	$\frac{1}{2}$
1	$\frac{1}{3}$
2	$\frac{1}{4}$
3	$\frac{1}{5}$
4	$\frac{1}{6}$

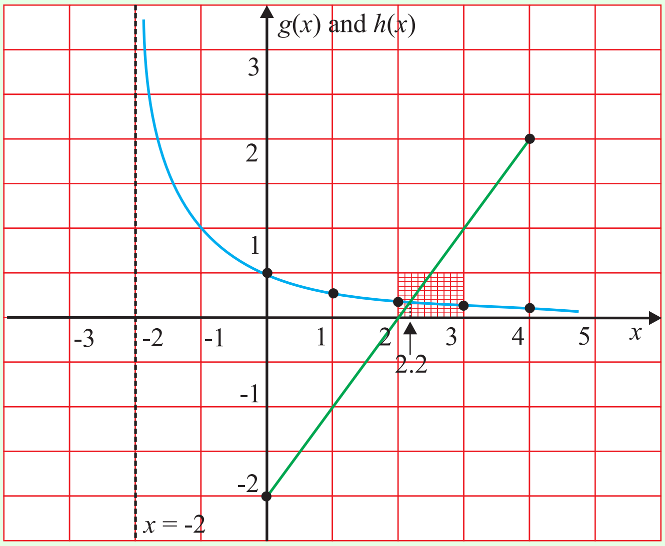


The graph of $h(x) = x - 2$ is a straight line graph. You just need to get 2 points on it to graph it. Choose the end points of the domain.

x	$h(x)$
0	–2
4	2

$x = 0 : h(0) = (0) - 2 = -2 \Rightarrow (0, -2)$ is a point on the graph.

$x = 4 : h(4) = (4) - 2 = 2 \Rightarrow (4, 2)$ is a point on the graph.



$x = 2.2$ is their point of intersection.

$$g(x) = h(x) \Rightarrow \frac{1}{x+2} = x - 2$$

$$\Rightarrow 1 = (x-2)(x+2) \Rightarrow 1 = x^2 - 4 \Rightarrow x^2 = 5$$

$$\therefore x = \sqrt{5}$$

$$\therefore \sqrt{5} \approx 2.2$$