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6 (a) Let $f(x) = 3x + k,\ x \in \mathbf{R}$.
If $f(5) = 0$, find the value of k.

(b) Let $g(x) = x^2 + bx + c,\ x \in \mathbf{R}$.
The solutions of $g(x) = 0$ are symmetrical about the line $x = 1$.
If $x = -3$ is one solution of $g(x) = 0$, find the other solution.
Find the value of b and the value of c.

(c) Let $f(x) = \dfrac{1}{x-2}$, for $x \in \mathbf{R}$ and $x \neq 2$.
Find the derivative of $f(x)$.
Tangents to $f(x)$ make an angle of 135° with the x axis.
Find the coordinates of the points on the curve of $f(x)$ at which this occurs.

SOLUTION

6 (a)

$f(x) = 3x + k$

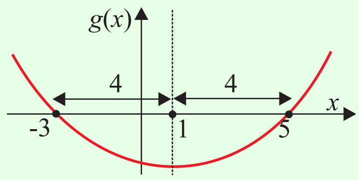
$f(5) = 0 \Rightarrow 3(5) + k = 0$

$\Rightarrow 15 + k = 0$

$\therefore k = -15$

6 (b)

$g(x) = x^2 + bx + c$ is a quadratic graph as shown to the right.
The line $x = 1$ is its axis of symmetry.
-3 is a solution (root) which is a place where the graph cuts the x-axis.
You can see from the diagram that the other solution is $x = 5$.
If you substitute solutions for x in the function $g(x)$, you get zero.



$x = -3: g(-3) = (-3)^2 + b(-3) + c = 0 \Rightarrow 9 - 3b + c = 0$

$\therefore -3b + c = -9 \ldots.(\mathbf{1})$

$x = 5: g(5) = (5)^2 + b(5) + c = 0 \Rightarrow 25 + 5b + c = 0$

$\therefore 5b + c = -25 \ldots.(\mathbf{2})$

Solve equation (**1**) and (**2**) simultaneously.

$$\boxed{\begin{aligned} -3b + c &= -9 \ldots.(\mathbf{1})(\times -1) \\ 5b + c &= -25 ..(\mathbf{2}) \end{aligned}} \rightarrow \boxed{\begin{aligned} 3b - c &= 9 \\ 5b + c &= -25 \\ \hline 8b \quad &= -16 \Rightarrow b = -2 \end{aligned}}$$

Substitute this value of b into Eqn. (**2**).

$5(-2) + c = -25 \Rightarrow -10 + c = -25 \Rightarrow c = -15$

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6 (c)

$y = f(x) = \frac{1}{x-2} = (x-2)^{-1}$

$\Rightarrow \frac{dy}{dx} = f'(x) = -1(x-2)^{-2}(1) = -\frac{1}{(x-2)^2}$

Move the power down in front of the bracket.
Take one away from the power.
Multiply by the differentiation of the inside of the bracket.

You are find the slope by getting the tan of the angle with the *x*-axis.

$m = \tan 135^\circ = -1$ [Using your calculator.]

$\boxed{\text{Slope } m = \tan\theta}$

STEPS: Finding points of contact given the slope, *m*, of the tangent

1. Find $\frac{dy}{dx}$.
2. Put $\frac{dy}{dx} = m$ and solve for *x*.
3. Find the corresponding *y* values.

1. $y = f(x) = \frac{1}{x-2}$

$\Rightarrow \frac{dy}{dx} = -\frac{1}{(x-2)^2}$

2. $\frac{dy}{dx} = -1 \Rightarrow -\frac{1}{(x-2)^2} = -1$

$\Rightarrow \frac{1}{(x-2)^2} = 1$

$\Rightarrow 1 = (x-2)^2$

$\Rightarrow \pm 1 = x - 2$

$\therefore x = 1, 3$

3. $x = 1: y = f(1) = \frac{1}{(1)-2} = \frac{1}{-1} = -1 \Rightarrow (1, -1)$ is a point of contact.

$x = 3: y = f(3) = \frac{1}{(3)-2} = \frac{1}{1} = 1 \Rightarrow (3, 1)$ is a point of contact.

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7 (a) Differentiate from first principles

$$3x-7$$

with respect to x.

(b) (i) Find $\dfrac{dy}{dx}$ when $y=\dfrac{2x}{4-x^2}$, for $x \in \mathbf{R}$ and $x \neq \pm 2$.

Show that $\dfrac{dy}{dx}>0$.

(ii) Differentiate $\left(x^5-\dfrac{1}{x^2}\right)^7$ with respect to x, $x \neq 0$.

(c) A stone is dropped from a height of 80 metres. Its height h metres above the ground after t seconds is given by

$$h=80-t^2.$$

Find

(i) its speed after t seconds

(ii) its speed after 2.5 seconds

(iii) the time it takes to fall the first 14.4 metres.

Solution

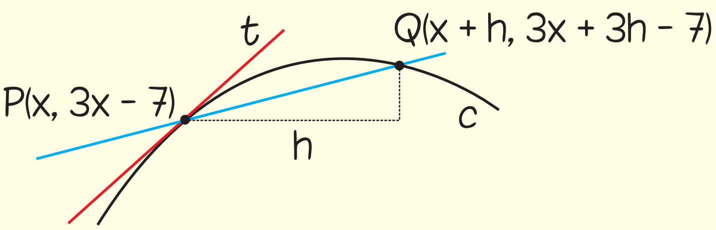
7 (a)

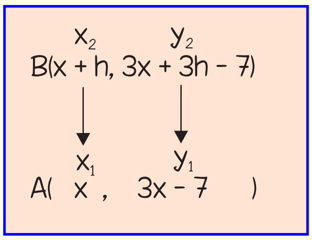
Steps

1. Find the y or $f(x)$ coordinates of P and Q.

2. Find the slope of PQ.

3. Find $\dfrac{dy}{dx}$ by finding the limit of the slope as h goes to zero.





$$m=\frac{y_2-y_1}{x_2-x_1}$$

$$y=3(x)-7$$
$$x=x: y=3x-7$$

$$y=3(x)-7$$
$$x=x+h: y=3(x+h)-7$$
$$=3x+3h-7$$

$$\text{Slope of } PQ=\frac{(3x+3h-7)-(3x-7)}{x+h-x}$$

$$=\frac{3x+3h-7-3x+7}{x+h-x}=\frac{3h}{h}=3$$

$$\frac{dy}{dx}=\lim_{h\to 0}(3)=3$$

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7 (b) (i)

$$u = 2x \Rightarrow \frac{du}{dx} = 2$$
$$v = 4 - x^2 \Rightarrow \frac{dv}{dx} = -2x$$

$$y = \frac{u}{v} \Rightarrow \frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$y = \frac{2x}{4 - x^2} \begin{array}{l} \leftarrow u \\ \leftarrow v \end{array}$$

$$\Rightarrow \frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2} = \frac{(4-x^2)(2) - 2x(-2x)}{(4-x^2)^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{8 - 2x^2 + 4x^2}{(4-x^2)^2} = \frac{2x^2 + 8}{(4-x^2)^2}$$

$\frac{dy}{dx} = \frac{2x^2+8}{(4-x^2)^2} > 0$ for both positive and negative values of x as when they are squared the answer will be positive.

7 (b) (ii)

Move the power down in front of the bracket.
Take one away from the power.
Multiply by the differentiation of the inside of the bracket.

$$u = x^5 - \frac{1}{x^2} = x^5 - x^{-2} \Rightarrow \frac{du}{dx} = 5x^4 + 2x^{-3} = 5x^4 + \frac{2}{x^3}$$

$$y = \left(x^5 - \frac{1}{x^2}\right)^7 \Rightarrow \frac{dy}{dx} = 7\left(x^5 - \frac{1}{x^2}\right)^6 \left(5x^4 + \frac{2}{x^3}\right)$$

$$\therefore \frac{dy}{dx} = \left(35x^4 + \frac{14}{x^3}\right)\left(x^5 - \frac{1}{x^2}\right)^6$$

7 (c) (i)

Draw up a s, v, a table as shown on the right.

$v = -2t$ m/s

$$v = \frac{ds}{dt}$$

$$a = \frac{dv}{dt}$$

7 (c) (ii)

Find its speed v after a time t = 2.5 seconds.

$v = -2t = -2 \times 2.5 = -5$ m/s

7 (c) (iii)

Find the time t it takes to fall a height h = 14.4 m.

$$h = 80 - t^2$$
$$v = \frac{dh}{dt} = -2t$$
$$a = \frac{dv}{dt} = -2$$

$h = 80 - t^2 \Rightarrow 14.4 = 80 - t^2$

$\Rightarrow t^2 = 80 - 14.4$

$\Rightarrow t^2 = 65.6$

$\therefore t = \sqrt{65.6} = 8.1$ s

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8 (a) Find $\frac{ds}{dt}$ when $s = 6t^2 - 3t + 7$.

(b) Let $f(x) = x^3 - 3x^2$, for $x \in \mathbf{R}$.

(i) Find $f'(x)$, the derivative of $f(x)$. Hence, calculate the coordinates of the local maximum and the local minimum of $f(x)$.

(ii) Draw the graph of

$$f(x) = x^3 - 3x^2$$

for $-1 \le x \le 3$.

(iii) Use your graph to estimate the values of x for which

$$f(x) + 2 = 0.$$

(iv) Use your graph to estimate the range of values of x for which

$$f'(x) < 0.$$

SOLUTION

8 (a)

$$\boxed{y = x^n \Rightarrow \frac{dy}{dx} = nx^{n-1}}$$

CONSTANT RULE: If y = Constant $\Rightarrow \frac{dy}{dx} = 0$

MULTIPLY BY A CONSTANT RULE: If $y = cu$, where c is a constant and u is a function of x, $\frac{dy}{dx} = c \times \frac{du}{dx}$.

$$s = 6t^2 - 3t + 7$$

$$\Rightarrow \frac{ds}{dt} = 6 \times 2t - 3 + 0 = 12t - 3$$

8 (b) (i)

$$y = f(x) = x^3 - 3x^2$$

$$\Rightarrow \frac{dy}{dx} = f'(x) = 3x^2 - 6x$$

$$\frac{dy}{dx} = 0 \Rightarrow 3x^2 - 6x = 0$$

$$\Rightarrow 3x(x-2) = 0$$

$$\therefore x = 0,\ 2$$

$$\boxed{\text{Turning Point: } \frac{dy}{dx} = 0}$$

$x = 0 : y = f(0) = (0)^3 - 3(0)^2 = 0 - 0 = 0 \Rightarrow (0,\ 0)$ is a local maximum.

$x = 2 : y = f(2) = (2)^3 - 3(2)^2 = 8 - 12 = -4 \Rightarrow (2,\ -4)$ is a local minimum.

[(0, 0) is the local maximum as its y coordinate is greater than the y coordinate of the other point.]

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8 (b) (ii)

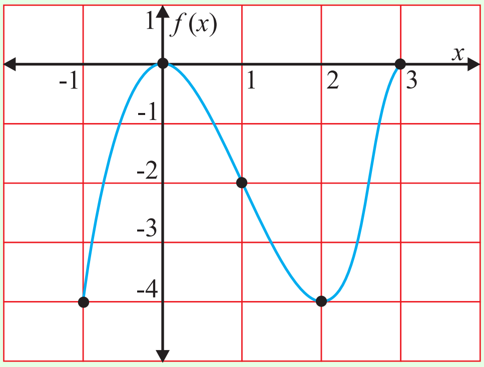
$x=-1: f(-1)=(-1)^3-3(-1)^2=-1-3\times 1=-1-3=-4 \Rightarrow (-1,-4)$ is a point.

$x=0: f(0)=(0)^3-3(0)^2=0-3\times 0=0-0=0 \Rightarrow (0, 0)$ is a point.

$x=1: f(1)=(1)^3-3(1)^2=1-3\times 1=1-3=-2 \Rightarrow (1,-2)$ is a point.

$x=2: f(2)=(2)^3-3(2)^2=8-3\times 4=8-12=-4 \Rightarrow (2,-4)$ is a point.

$x=3: f(3)=(3)^3-3(3)^2=27-3\times 9=27-27=0 \Rightarrow (3, 0)$ is a point.



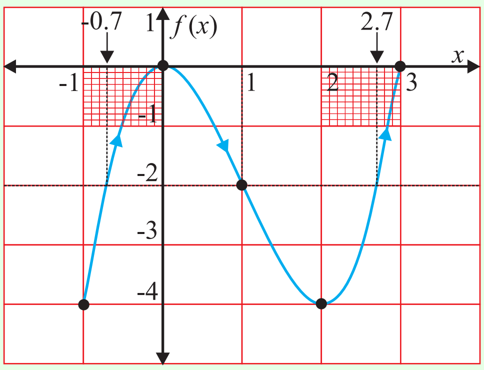
x	$f(x)$
–1	–4
0	0
1	–2
2	–4
3	0

8 (b) (iii)

$f(x)+2=0 \Rightarrow f(x)=-2$

Go to -2 on the $f(x)$ axis. Draw a line straight across until it meets the graph. Read off the x values.

$\therefore x=-0.7, 1, 2.7$



8 (b) (iv)

$\boxed{f'(x)<0\text{: Curve is decreasing; } f'(x)>0\text{: Curve is increasing.}}$

You can see the graph is decreasing from values of x from 0 to 2.

$\therefore\ f'(x)<0 \Rightarrow 0<x<2$