LC 2014: Paper 1

Question 2 (25 marks)

Question 2 (a)

Use the conjugate root theoreom to find the other root. Therefore, if z_1 is one of the roots, the conjugate of z_1 is also a root. Call this root z_2.

$$z_1 = 1 - 2i \Rightarrow z_2 = 1 + 2i$$

Forming a quadratic equation from its roots.

$\mathbf{S}: 1 - 2i + 1 + 2i = 2$

$\mathbf{P}: (1 - 2i)(1 + 2i) = 5$

$z^2 - 2z + 5 = 0$

The cubic equation is made up of a quadratic equation multiplied by a linear equation.

$2z^3 - 7z^2 + 16z - 15 = (z^2 - 2z + 5)(\qquad) = 0$

$\mathbf{2z^3} - 7z^2 + 16z - 15 = (\mathbf{z^2} - 2z + 5)(\mathbf{2z} \qquad) = 0$

$2z^3 - 7z^2 + 16z \mathbf{- 15} = (z^2 - 2z + \mathbf{5})(2z \mathbf{- 3}) = 0$

$(2z - 3) = 0 \Rightarrow z = \frac{3}{2}$

Roots: $z_2 = 1 + 2i,\ z_3 = \frac{3}{2}$

If all the coefficients of a quadratic equation are real, then the roots are both real or are complex conjugates of each other, and vice versa.

Conjugate of z is represented by $\bar{z}$

$$z = a + bi \Rightarrow \bar{z} = a - bi$$

Forming a quadratic equation from its roots.

$$z^2 - \mathbf{S}z + \mathbf{P} = 0$$

S: Sum of the roots
P: Product of the roots

Cubic = Quadratic × Linear
The first term in the cubic equals the first term in the quadratic multiplied by the first term in the linear.
Similarly, the last term in the cubic equals the last term in the quadratic multiplied by the last term in the linear

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Other methods (not as effiicent):
$1 - 2i$ is a root $\Rightarrow (z - 1 + 2i)$ is a linear factor
$1 + 2i$ is a root $\Rightarrow (z - 1 - 2i)$ is a linear factor
Quadratic factor: $(z - 1 + 2i)(z - 1 - 2i)$

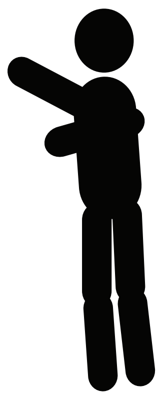
$(z - 1 + 2i)(z - 1 - 2i)$

$= z^2 - z - 2iz - z + 1 + 2i + 2iz - 2i - 4i^2 \quad [i^2 = -1]$

$= z^2 - 2z + 1 + 4$

$= z^2 - 2z + 5$

Author's note: The best technique for finding the roots of the cubic equation are shown at the beginning of the solution. Only 5 marks are allocated to this so I really want my answers within the minute.



$$\begin{array}{r|l} & 2z-3 \\ z^2-2z+5 & 2z^3-7z^2+16z-15 \\ & \mp 2z^3 \pm 4z^2 \mp 10z \\ & \quad -3z^2+6z-15 \\ & \quad \pm 3z^2 \mp 6z \pm 15 \\ & \quad\quad 0 \end{array}$$

$(2z-3)=0 \Rightarrow z=\frac{3}{2}$

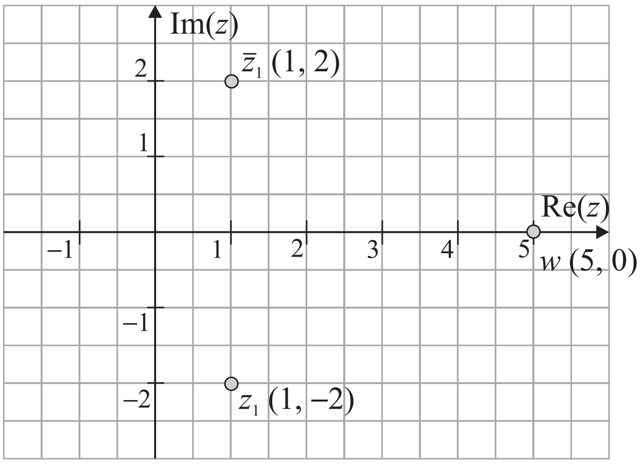
Roots: $z_2 = 1+2i,\ z_3=\frac{3}{2}$

Marking Scheme Notes
Question 2 (a) [Scale 5D (0, 2, 3, 4, 5)]
2: • Identifies another root
• Forms an equation
3: • Works with correct quadratic factor
• Indicates division of quadratic into cubic
4: • Finds third factor

Question 2 (b) (i)

Formula: Complex Numbers
Congugates $\bar{z}$
$z=a+bi \Rightarrow \bar{z}=a-bi$

$z_1 = 1-2i$
$\bar{z}_1 = 1+2i$
$w = z_1\bar{z}_1 = (1-2i)(1+2i) = 5 = 5+0i$

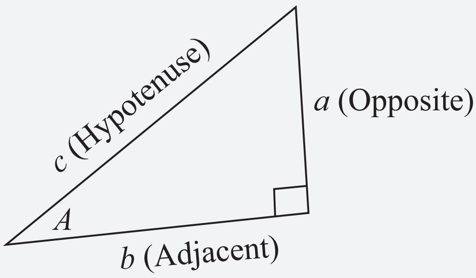


Marking Scheme Notes
Question 2 (b) (i) [Scale 10C (0, 5, 7, 10)]
5: • Plots one point correctly
• Finds $\bar{z}_1$
7: • Points plotted but not labelled or labelled incorrectly
• Two points plotted and labelled
• Calculates w

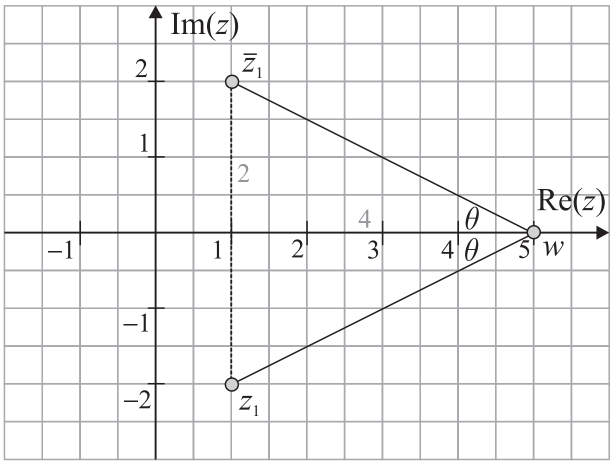
Question 2 (b) (ii)

Method 1: Divide triangle into two right-angled triangles and use tan.

Formulae and Tables Book
Trigonometry: Right-angled triangle [page 16]



$\sin A = \frac{a}{c},\ \cos A = \frac{b}{c},\ \tan A = \frac{a}{b}$



$$\tan\theta = \tfrac{2}{4} = \tfrac{1}{2} \Rightarrow \theta = 26\cdot6^{\circ}$$
$$|\angle \bar{z}_1 w z_1| = 2\times 26\cdot6^{\circ} \approx 53^{\circ}$$

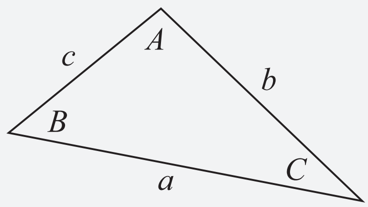
Method 2: Use Cosine rule (not as efficient)

$$|z_1 w| = \sqrt{(0-(-2))^2 + (5-1)^2} = \sqrt{20}$$
$$|\bar{z}_1 w| = \sqrt{20}$$
$$|\bar{z}_1 z_1| = 4$$

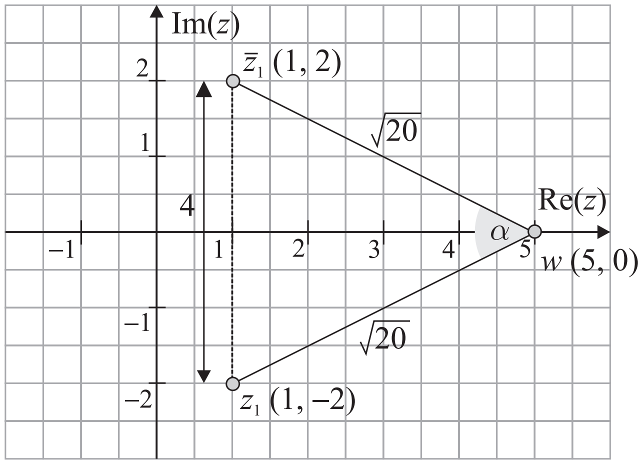
Formula: Complex Numbers
Modulus $|z|$

$$|z| = |a+bi| = \sqrt{a^2+b^2}$$

Formulae and Tables Book
Trigonometry of the triangle:
Cosine rule [page 16]



$$a^2 = b^2 + c^2 - 2bc\cos A$$



$$4^2 = (\sqrt{20})^2 + (\sqrt{20})^2 - 2(\sqrt{20})(\sqrt{20})\cos\alpha$$
$$16 = 20 + 20 - 40\cos\alpha$$
$$40\cos\alpha = 24$$
$$\therefore \cos\alpha = \frac{24}{40} = 0\cdot6 \Rightarrow \alpha = \cos^{-1}(0\cdot6) = 53\cdot13^{\circ} \approx 53^{\circ}$$

Marking Scheme Notes
Question 2 (b) (ii) [Scale 10C (0, 5, 7, 10)]

5:
- Length of any one side of triangle calculated correctly
- Correct definition of trig ratio
- Correct cos rule
- Recognises the half-angle

7:
- cos value calculated but angle not found
- tan value of half-angle calculated